3 Introducing Logical Vocabulary

Making Reason Relations Logical

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In this chapter, we introduce a broadly proof-theoretic way to codify and theorize open reason relations among sentences of a language that contains logical vocabulary. We interpret this proof-theoretic formalism in a pragmatic-normative way. Throughout this book, we restrict ourselves to the sentential level, as things turn out to be fascinating and complex enough at that level. Thus, we are mainly concerned with negation ($\neg$), the conditional ($\rightarrow$), disjunction ($\vee$), and conjunction ($\wedge$). In this chapter, we present a sequent calculus treatment of these connectives that is almost entirely standard and conventional. What is exciting about this treatment, however, is that it allows us to capture, in a formally rigorous way, the ideas of logical expressivism that we have presented in the previous chapters, and this includes capturing open reason relations in which topological closure as well as explicitation closure fail. In later chapters, we will often return to the logical systems that we introduce in this chapter. One of the chief accomplishments of this book lies in the fact that we show how these logical systems appear and emerge naturally from different perspectives, including the perspective of truth-maker theory in the next chapter and the more abstract perspective of implication-space semantics in Chapter Five.

Before turning to more formal and technical matters, let us recapitulate where we are in our philosophical project. In the first two chapters, we have introduced the idea of reason relations according to a pragmatics-first approach, as well as the thesis of logical expressivism. To repeat, we start with a discursive practice of giving reasons for and against claims. This practice is constrained by reason relations of implication and incompatibility. In particular, the sentences in a set Γ are jointly a reason for the sentence A if and only if Γ implies A, which we write as $\Gamma \mathrel{|\!\sim} A$. And the sentences in Γ are jointly a reason against the sentence A if and only if Γ is incompatible with A, which we write as $\Gamma \# A$. We encode both relations in one single relation by stipulating that $\Gamma, A \mathrel{|\!\sim}$ (notice the empty right side) just in case $\Gamma \# A$. According to logical expressivism, it is the characteristic function of logical vocabulary to make such reason relations explicit.

The sense of "reason for" and "reason against" in this book is emphatically not the sense in which a reason for or against a claim makes it the case that one ought to accept or reject the claim, respectively. For, what one ought to do may be to reject the reason, as Harman (1986) has stressed in his work on the normativity of logic. Rather, if one accepts or rejects something for a (operative) reason, then this acceptance or rejection is performed for good reason only if the (operative) reason is indeed a reason (in our sense) for or against, respectively, what one accepts or rejects. Thus, if one accepts A for the reason Γ, then that Γ implies A is a necessary but not a sufficient condition for one's accepting A being done for good reason.

We have stressed that not all good reasons are logically good reasons, and we call the mistake of thinking otherwise "logicism about reasons." Correspondingly, our implication relation includes more than just logical implications, and our incompatibility relation includes more than just logical incompatibilities. We denote classical propositional consequence by $\vdash_{\overline{CL}}$. So we know that $\mid\!\sim$ is not identical to $\vdash_{\overline{CL}}$. Indeed, we have argued that $\mid\!\sim$ ought to differ from classical consequence in fundamental and radical ways. In particular, it should be nonmonotonic and nontransitive. For, our aim is to offer a treatment of logical vocabulary as being able to make explicit open reason relations. Nevertheless, one might plausibly claim that, while not all good implications are logically good implications, all implications of classical logic are indeed good implications, so that $\vdash_{\overline{CL}} \subset \mid\!\sim$. This classical subset of all implications obeys, of course, traditional structural principles, and this turns out to be the case for our theory below. We are thus not offering a nonmonotonic logic but rather an account of consequence and a logic that can codify generally nonmonotonic material implications as well as monotonic logical implications.

In contrast to standard approaches to nonmonotonic logic, we do not start with a familiar logical consequence relation, like $\vdash_{\overline{CL}}$, and then try to extend it to a nonmonotonic consequence relation. Rather, we start with a nonmonotonic consequence relation over atomic sentences, which we then extend to a logically complex language in such a way that the result includes classical consequence. The way in which we do this is to take the implications among atomic sentences and to treat them as axioms in a sequent calculus. Moreover, unlike most extant approaches to nonmonotonic consequence, we treat implications and incompatibilities as equally subject to failures of monotonicity.[1] Thus, not only can implications be defeated by adding further premises but also incompatibilities can be "cured" by adding further claims.[2]

The chapter is structured as follows: in the first subsection, we introduce the basic sequent calculus NMMS, which illustrates how logical vocabulary can be elaborated from any autonomous base vocabulary. In the second subsection, we explain two different senses in which the logical vocabulary introduced by NMMS can make reason relations explicit, including open reason relations. Thus, the first two subsections together present a treatment of logical vocabulary as vocabulary that is universally LX, that is, vocabulary that can be elaborated from any autonomous base vocabulary and can make explicit the reason relations among sentences of any base vocabulary from which it is elaborated. In the second subsection, we also formulate a variant of our sequent calculus that works for consequence relations in which Contraction fails. And in the third subsection, we present more adjustments and additions to our sequent calculus. In particular, we show how we can introduce logical vocabulary that makes explicit local structural features of reason relations. The fourth section concludes; and the appendix provides proofs and lemmas that we omit in the main text, in order to make the main text more accessible.

3.1 Sequent Calculi for Logical Expressivism

The brilliant idea behind Gentzen's (1934) invention of sequent calculus is to encode entire implications, rather than just sentences, in objects that are manipulated in a calculus. These objects are called "sequents," and we will write them, for example, like this: $\Gamma \succ \Delta$, where Γ is the set of the premises and Δ is a conclusion or a set of conclusions.[3] In a sequent calculus, instead of deriving sentences from sentences, as in natural deduction or axiomatic systems, we derive sequents from sequents, and the thought is that sequents that are derivable correspond to good implications. This idea fits well with the aims of logical expressivism. For, according to logical expressivism, the proper starting point for the formulation of an account of consequence that includes logical consequence is a set of implications that hold among nonlogical, that is, atomic, sentences. Such implications can serve as the input or axioms of a sequent calculus, as will become clearer below.

Moreover, according to the semantic inferentialism that informs our larger project, such implications and incompatibilities among declarative atomic sentences articulate the conceptual roles played by these sentences. In this sense, the axioms of our sequent calculi articulate the conceptual roles of the nonlogical vocabulary. A sequent calculus extends these conceptual roles of atomic sentences to a consequence relation that also articulates the conceptual roles of logically complex sentences.

3.1.1 How Many Conclusions?

Before moving on, we should pause to note that, for different logics, Gentzen uses two different kinds of sequents, namely sequents that can have at most one conclusion and sequents that can have many conclusions. This is often put by saying that we can work in a Set-Formula framework or in a Set-Set framework, depending on whether the conclusion is a single formula or a set of formulae. And this can be generalized further, in one dimension, by working with lists or multi-sets or even tree-structures instead of sets; and it can be generalized, in another dimension, by working with sequents that don't have two sides but one, or three, or any other number of sides.[4] These differences have important consequences for the logics that are formulated in a sequent calculus. It is, for instance, one of Gentzen's surprising results that his sequent rules yield intuitionistic logic in the Set-Formula framework, but the very same sequent rules yield classical logic in the Set-Set framework.

In the previous chapters, we have often talked as if $\mid\!\sim$ is always flanked by a set of premises on the left and a single conclusion on the right. Thus, we have seemingly used a Set-Formula framework. We did this to avoid confusions that might arise from the fact that it is not immediately clear how to think philosophically about sequents with multiple conclusions. After all, a good implication seems to correspond to a good argument or inference, and such arguments or inferences have one or more premises but just one single conclusion. Similarly, it seems that we give reasons for or against single claims. Note, however, that we said that Γ is a reason (in our technical sense) for *A* if and only if commitment to accept all the sentences in Γ precludes entitlement to reject *A*.[5] And Γ is a reason against *A* if and only if commitment to accept all the sentences in Γ precludes entitlement to accept *A*. These notions can readily be generalized to allow for multiple conclusions, namely as follows: Γ is a reason for Δ if and only if commitment to accept all the sentences in Γ precludes entitlement to reject all the sentences in Δ. And Γ is a reason against Δ if and only if commitment to accept all the sentences in Γ precludes entitlement to accept all the sentences in Δ.

This way of understanding multiple conclusion sequents is a variant of the view developed by Restall (2005). He calls collections of assertions and denials "positions," and he writes the position in which everything in Γ is asserted and everything in Δ is denied as [Γ : Δ]. Restall then suggests that we can understand the claim that Γ implies Δ as the claim that it is normatively improper or "out-of-bounds"—as Restall says—to assert everything in Γ and deny everything in Δ, that is, the position [Γ : Δ] is out-of-bounds. If we think that a position is out-of-bounds just in case one cannot be entitled to all of the commitments in the position, then our suggestion and Restall's suggestion coincide. We take these to be equivalent ways to understand implication or consequence in pragmatic-normative terms, and we will sometimes refer to this understanding generically as "normative bilateralism."[6]

We will usually work with sequents with multiple conclusions, thus relying on the interpretation of such sequents just given.[7] We usually work with sets of premises and conclusions, but we occasionally also use multi-sets. So, unless specified otherwise, our sequents will be of the form $\Gamma \succ \Delta$, where Γ and Δ are sets of sentences (availing ourselves of the usual ways to omit set-theoretic notation in sequents). And we think correspondingly about implication or consequence as also allowing for multiple conclusions, so that we can write $\Gamma \mid\!\sim \Delta$ to say that the set Γ implies the set Δ or, equivalently, the set Δ is a consequence of the set Γ. As explained above, we will write $\Gamma, \Delta \mid\!\sim$ to say that Γ and Δ are incompatible, and so the sentences in one of these sets are jointly a reason against those in the other set (jointly).

3.1.2 Adding Logical Vocabulary

As explained in the previous chapters, we use the term "vocabulary" to mean a collection of expressions together with the reason relations that hold among (sets of) them. And we formulated the core thesis of logical expressivism as the claim that logical vocabulary is, at least as an ideal, vocabulary that is universally LX, that is, it can be *el*aborated from and is *ex*plicative of any autonomous base vocabulary. More precisely, some vocabulary is universally LX just in case, for any base vocabulary that is sufficient to have a discursive practice of giving and asking for reasons for and against claims, (i) the reason relations among sentences that include bits of the universally LX vocabulary can be elaborated from the reason relations that hold among sentences of the base vocabulary, and (ii) the LX vocabulary allows its users to make explicit the reason relations that hold in the base vocabulary (and in the thus extended vocabulary).

In order to formulate a logic in accordance with this idea of logical expressivism, we must consider two questions: (a) What does it mean to elaborate reason relations among sentences featuring logical vocabulary from reason relations among sentences that do not feature any logical vocabulary? (b) What does it mean that sentences featuring logical vocabulary allow us to make explicit reason relations? In the remainder of this section, we address the first of these two questions by formulating an answer in terms of a sequent calculus. We turn to the second question in the following section.

We start with a base or base vocabulary $\mathfrak{B}$, which we assume to be autonomous, in the sense of being sufficient to have a discursive practice of giving and asking for reasons. We will think of the lexicon of our nonlogical, base vocabulary as a countable set of atomic sentences. Let us call this atomic language $\mathfrak{L}_{\mathfrak{B}}$. We encode the relations of reasons-for and reasons-against among sentences of $\mathfrak{L}_{\mathfrak{B}}$ in a single consequence relation $\mid\!\sim_{\mathfrak{B}}$, as explained above. Hence, $\Gamma \mid\!\sim_{\mathfrak{B}} \Delta$ holds just in case the sentences in Γ are, jointly, a reason for those in Δ. And $\Gamma, A \mid\!\sim_{\mathfrak{B}}$ holds just in case the sentences in Γ are jointly a reason against A.[8]

Definition 1 (Material Base). A material base, $\mathfrak{B}$, is an atomic language, $\mathfrak{L}_{\mathfrak{B}}$, and a base consequence relation, $\mid\!\sim_{\mathfrak{B}}$, between subsets of this language $\mid\!\sim_{\mathfrak{B}} \subseteq \mathcal{P}(\mathfrak{L}_{\mathfrak{B}}) \times \mathcal{P}(\mathfrak{L}_{\mathfrak{B}})$. A base consequence relation obeys Containment just in case $\langle \Gamma, \Delta \rangle \in \mid\!\sim_{\mathfrak{B}}$ whenever $\Gamma \cap \Delta \neq \emptyset$.

In order to extend our base vocabulary so as to include logical vocabulary, we must do two things. First, we must add logical expressions to the lexicon $\mathfrak{L}_{\mathfrak{B}}$. Second, we must specify the reason relations among our thus extended lexicon. We do the first by adding the logical expressions denoted by $\neg$, $\rightarrow$, $\wedge$, and $\vee$ to $\mathfrak{L}_{\mathfrak{B}}$ in the usual way. We call this extended lexicon $\mathfrak{L}$, and, to be precise, we can define it recursively by saying: firstly, all the sentences in $\mathfrak{L}_{\mathfrak{B}}$ are in $\mathfrak{L}$; secondly, if ϕ and ψ are any sentences in $\mathfrak{L}$, then $\neg\phi$, $\phi \rightarrow \psi$, $\phi \wedge \psi$, and $\phi \vee \psi$ are also in $\mathfrak{L}$; and lastly nothing else is in $\mathfrak{L}$.

We turn to a sequent calculus to perform our second task, namely to specify the reason relations among the sentences of our extended lexicon.[9] In formal terms, we have to specify a consequence relation $\mid\!\sim$ over sets of sentences of $\mathfrak{L}$, starting from the consequence relation $\mid\!\sim_{\mathfrak{B}}$ over $\mathfrak{L}_{\mathfrak{B}}$. Now, while sequent calculi can be formulated in many different ways, one can often formulate them by using sequents with only atomic sentences as axioms. That is what we do here. Where standard approaches use atomic sequents with certain formal properties (such as those with a non-empty intersection of the two sets, that is, instances of Containment), however, we will simply use our entire base consequence relation $\mid\!\sim_{\mathfrak{B}}$. So, calling our sequent calculus NMMS (for *n*on-*m*onotonic *m*ulti-*s*uccedent), we stipulate that $\Gamma \succ \Delta$ is an axiom of NMMS if and only if $\Gamma \mid\!\sim_{\mathfrak{B}} \Delta$. We write $\mathsf{NMMS}_{\mathfrak{B}}$ if we want to be clear what the base vocabulary is.

We can now use sequent rules to derive sequents with logically complex sentences from our axioms. Sequent rules are usually written as one or more schematic sequents, the top sequents, above a horizontal line and one sequent below that line, the bottom sequent. The sentences that occur unchanged in top sequents and bottom sequents are called the "context," and the sentences that occur, as free-standing sentences, only above or below the line are called the "active formulae" of the rule; sometimes the active sentence below the line is called the "principal formula." To see how this works, let's look at NMMS:

Axioms of NMMS:
Ax1: $\Gamma \succ \Delta$ is an axiom if and only if $\Gamma \mathrel{|\!\!\!\sim_{\mathfrak{B}}} \Delta$.

Rules of NMMS:

$$\frac{\Gamma \succ \Delta, A \qquad B, \Gamma \succ \Delta \qquad B, \Gamma \succ \Delta, A}{\Gamma, A \to B \succ \Delta}\ [\mathrm{L}\!\to] \qquad \frac{\Gamma, A \succ B, \Delta}{\Gamma \succ A \to B, \Delta}\ [\mathrm{R}\!\to]$$

$$\frac{\Gamma \succ \Delta, A}{\Gamma, \neg A \succ \Delta}\ [\mathrm{L}\neg] \qquad \frac{\Gamma, A \succ \Delta}{\Gamma \succ \Delta, \neg A}\ [\mathrm{R}\neg]$$

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \wedge B \succ \Delta}\ [\mathrm{L}\wedge] \qquad \frac{\Gamma \succ \Delta, A \qquad \Gamma \succ \Delta, B \qquad \Gamma \succ \Delta, A, B}{\Gamma \succ \Delta, A \wedge B}\ [\mathrm{R}\wedge]$$

$$\frac{\Gamma, A \succ \Delta \qquad \Gamma, B \succ \Delta \qquad \Gamma, A, B \succ \Delta}{\Gamma, A \vee B \succ \Delta}\ [\mathrm{L}\vee] \qquad \frac{\Gamma \succ \Delta, A, B}{\Gamma \succ \Delta, A \vee B}\ [\mathrm{R}\vee]$$

The rules of NMMS are slight variations on rules that are familiar as so-called "Ketonen-style rules" (see Negri et al., 2008; Humberstone, 2011; Bimbó, 2015). The only difference is that the rules that have more than one top sequent have exactly two top sequents in Ketonen-style rules, but they have three top sequents in NMMS. The third top sequent is such that all the active formulae in the other top sequents occur in the third top sequent, and they occur on the same side as in the other top sequent in which they occur. The reason to include the third top sequents in these rules is that they ensure that the number of copies of a sentence on the left or the right of the sequent arrow does not make any difference to which sequents are derivable from which other sequents. The difference matters only when Monotonicity fails. For, Monotonicity allows one to derive the unfamiliar third top sequent from either one of the two traditional top sequents of the Ketonen-style rules.

As an example of what these rules say, the rule [R→] says that if we have a sequent in which a sentence A is on the left side and a sentence B is on the right side, then we can derive the sequent that is like the first except that instead of A on the left and B on the right it includes the sentence $A \to B$ on the right side.[10] Thus, the rule [R→] immediately ensures that one direction of the Deduction-Detachment condition holds. And explanations of the other rules can be given in an analogous way.[11]

A sequent is derivable just in case there is a tree of applications of these sequent rules in which the sequent in question is at the root and every sequent that sits at a leaf of a tree is an axiom. In the following example of such a proof tree, the root is $\succ ((p \wedge r) \vee s) \rightarrow q$ and the leaves are $p, r \succ q$ and $s \succ q$ and $p, r, s \succ q$.

$$\dfrac{\dfrac{\dfrac{p, r \succ q}{p \wedge r \succ q}[L\wedge] \quad s \succ q \quad \dfrac{p, r, s \succ q}{p \wedge r, s \succ q}[L\wedge]}{(p \wedge r) \vee s \succ q}[L\vee]}{\succ ((p \wedge r) \vee s) \rightarrow q}[R\rightarrow]$$

The sequent calculus NMMS is a way to perform our second task, namely the task of specifying the reason relations among the sentences of our extended lexicon, because we can now say that, in our extended language $\mathfrak{L}$, a set of sentences Γ implies a set of sentences Δ just in case the sequent $\Gamma \succ \Delta$ is derivable in NMMS. That is, $\Gamma \mathrel{|\!\sim} \Delta$ if and only if $\Gamma \succ \Delta$ is derivable in NMMS. In this way, we can take any base vocabulary $\mathfrak{B}$, consisting of a set of sentences $\mathfrak{L}_{\mathfrak{B}}$ and a consequence relation over them $\mathrel{|\!\sim}_{\mathfrak{B}}$, and extend it to include logical vocabulary, by moving to the language $\mathfrak{L}$ and consequence relation $\mathrel{|\!\sim}$ over the extended language $\mathfrak{L}$. The upshot of the example proof tree above, for example, is that if $p, r \mathrel{|\!\sim}_{\mathfrak{B}} q$ and $s \mathrel{|\!\sim}_{\mathfrak{B}} q$ and $p, r, s \mathrel{|\!\sim}_{\mathfrak{B}} q$, then $\mathrel{|\!\sim} ((p \wedge r) \vee s) \rightarrow q$. We call the reason relations that hold among sentences in our logically extended lexicon the "logically extended reason relations."

Note that NMMS does not include any of the following rules which are often used in sequent calculi and are among the traditional structural principles (together with Permutation, which we take for granted throughout this book), where the first rule allows us to derive a bottom sequent without using any top sequents:

$$\dfrac{}{A \succ A}[\text{Reflexivity}]$$

$$\dfrac{\Gamma, A, A \succ \Delta}{\Gamma, A \succ \Delta}[\text{L-Contraction}] \qquad \dfrac{\Gamma \succ A, A, \Delta}{\Gamma \succ A, \Delta}[\text{R-Contraction}]$$

$$\dfrac{\Gamma \succ \Delta}{\Gamma, \Theta \succ \Delta, \Lambda}[\text{Weakening}] \qquad \dfrac{\Gamma \succ \Delta, A \quad A, \Theta \succ \Lambda}{\Gamma, \Theta \succ \Delta, \Lambda}[\text{Mixed-Cut}]$$

The reason why NMMS does not include [Reflexivity], [L-Contraction], and [R-Contraction] as rules is very different from the reason why NMMS does not include [Weakening] and [Mixed-Cut]. It doesn't include the former rules because, given a base that obeys Containment, these rules are not necessary; adding them to NMMS does not change what is derivable in NMMS. All the sequents derivable by [Reflexivity] are already axioms if the sentence in the sequent is atomic, and one can derive them using the rules of NMMS if the sentence is logically complex. The sequent $p \wedge q \succ p \wedge q$, for instance, is derivable from $p, q \succ p$ and $p, q \succ q$ and $p, q \succ p, q$, by using [R∧] and [L∧]; and it is easy to show that this is always possible (see Appendix, Proposition 24). The two contraction rules are not rules of NMMS because we are working with sets of sentences on the left and the right side of sequents. Hence, the top sequents and the bottom sequents of the contraction rules are identical (although this will not be true for the calculus $\mathsf{NMMS}^{\backslash ctr}$ below). Thus, the rules for Reflexivity and Contraction are not included because they would be redundant and we prefer not to list them as rules.

The reason why NMMS does not include [Weakening] and [Mixed-Cut] is very different. The [Weakening] rule encodes the constraint that reason relations must be monotonic, and the [Mixed-Cut] rule is a generalized version of the idea that reason relations must be transitive. These rules fail and have counterexamples in NMMS when we codify open reason relations, assuming that our sequent rules ought to extend the base consequence relation in a conservative way, that is, assuming that if all the sentences in Γ and Δ are in the base lexicon, then $\Gamma \mid\!\sim \Delta$ just in case $\Gamma \mid\!\sim_{\mathfrak{B}} \Delta$ (see below). For [Weakening], a base consequence relation that is such that $p \mid\!\sim_{\mathfrak{B}} q$ but $p, r \not\mid\!\sim_{\mathfrak{B}} q$ is a counterexample. And [Mixed-Cut] actually implies [Weakening], given Containment, and so it yields to the same counterexample. To see this, suppose again that $p \mid\!\sim_{\mathfrak{B}} q$. Now, by Containment, $p, q, r, \mid\!\sim_{\mathfrak{B}} q$. And so [Mixed-Cut] would allow us to prove that $p, r \mid\!\sim q$. Hence, we must reject [Weakening] and [Mixed-Cut] and allow that they can fail. These failures make the consequence relations that we shall study substructural: some of the traditional structural principles fail in them.

Returning to the thesis of logical expressivism, note that vocabulary that can be introduced into a language in the way in which we have just introduced the vocabulary of propositional logic can be elaborated from any autonomous base vocabulary, and it thus meets the first part of the condition of logicality suggested by logical expressivism, that is, the first part of the condition that logical vocabulary must be universally LX. For, someone who can use any base language, in accordance with its reason relations, already knows how to do everything they need to know how to do in order to acquire the ability to use the vocabulary introduced in this subsection, in accordance with its reason relations. All that such a person needs to do is to leverage their sensitivity to reason relations among atomic sentences to yield a sensitivity to reason relations among sentences that include logical vocabulary in accordance with the sequent rules above.[12] And we can leave it open what exactly the relevant kind of sensitivity to reason relations is that a speaker must have in order to be a competent speaker of some base language. As long as this ability can be transformed in a way that corresponds to the sequent rules above, all that a speaker needs to do in order to acquire the ability to use logical vocabulary is to undergo such a transformation. Crucially, this does not require any new experience of, or familiarity with, objects or properties that are novel to the speaker. In this sense, the potentiality of understanding and using logical vocabulary is contained in the ability to understand and use any autonomous vocabulary whatsoever. This is the first part of the idea that logical vocabulary is vocabulary that is universally LX, and we have now explained how extending a base vocabulary by means of a sequent calculus in the way sketched above meets this first criterion of logicality.

3.1.3 Some Familiar Features of Logical Vocabulary

The result of adding logical vocabulary and specifying its reason relations by NMMS is a radical departure from standard approaches to logic in many ways, but it is also conventional in other ways. So let us highlight some properties of reason relations that are extended by NMMS. We start, in this subsection, with features that our theory shares with many familiar sequent calculi treatments of logical vocabulary, and we will discuss the radical and novel aspects of our theory in the next subsection.

As already intimated, the rules of NMMS are slight variations on rules that are familiar as so-called "Ketonen-style rules" (see Negri et al., 2008; Humberstone, 2011; Bimbó, 2015). And the rules of NMMS share the key virtues of the Ketonen-style rules, in particular the following three: First, the rules allow us to derive all classically valid sequents without the help of structural rules if the base consequence relation obeys Containment. Second, the rules are such that every sentence that occurs in any of the top sequents of a rule application also occurs, either as a free-standing sentence or as a subformula of a sentence, in the bottom sequent,[13] and the top sequents are always all at most as logically complex as the bottom sequent. Third, the rules are invertible, which means that if the bottom sequent of an application of a rule is derivable, then all the top sequents are also derivable. Let us explain these three features in a bit more detail.

Regarding the first familiar feature of NMMS, note that if the base consequence relation obeys Containment, then any sequent of the form $\Gamma, p \succ p, \Delta$, of sentences in the base lexicon, is an axiom. And it can easily be shown that closing sequents of this form under the rules of NMMS yields exactly the relation of classical propositional consequence, over the extended lexicon. Since adding more sequents as axioms does not affect the derivations of the classically valid sequents, closing any base consequence relation that obeys Containment under the rules of NMMS includes all the implications of classical propositional logic.

Fact 2. *If a base obeys Containment, then any implication among sentences of $\mathfrak{L}$ that holds in classical propositional logic also holds in* $\mathrel{|\!\sim}$*, that is,* $\vdash_{CL} \subseteq \mathrel{|\!\sim}$*. (Appendix, Proposition 25)*

In other words, insofar as everything implies itself (with arbitrary contexts), we agree with classical logicians regarding every classically valid implication that it is indeed valid. And since almost all non-classical logics are strictly weaker than classical logic,[14] in that they deny the validity of some implication that is deemed valid by classical logic, our position can seem very classical at this point. Indeed, if we say that the "narrowly logical part" of the logically extended consequence relation is the part that can be derived in NMMS from just the instances of Containment, then the narrowly logical part of our theory just is classical logic. Equivalently, the logically good reason relations are those that hold in the logical extensions of all base vocabularies that obey Containment. Of course, all of the structural principles hold in the narrowly logical part of our logically extended consequence relation; in particular, this narrowly logical part obeys Monotonicity and Transitivity. So, while the logically extended consequence relation of a base that obeys Containment will typically be nonmonotonic and nontransitive, the narrowly logical part of this consequence relation will be monotonic and transitive. In this sense, we are offering a fully structural logic for the purpose of codifying open reason relations, that is, reason relations that are substructural. Thus, if we identify a logic with the narrowly logical part of a consequence relation, then we are not offering a nonmonotonic logic, but rather a monotonic—indeed classical—logic whose logical vocabulary allows us to make explicit nonmonotonic consequence relations.

The consequence relations defined by NMMS are substructural in the sense that Monotonicity and Transitivity can fail in them, namely if they fail in the base consequence relations that are extended by the rules of NMMS. Sequent rules for the connectives that are equivalent in the context of structural rules often differ in important respects in substructural settings. The so-called multiplicative and additive rules of linear logic are perhaps the most common rules for conjunction and disjunction in a substructural setting, thus yielding two conjunctions and two disjunctions. Our (quasi) Ketonen-style sequent rules use, in effect, additive rules on one side and multiplicative rules on the other side. If our base consequence relations obey Containment, then the differences between these rules only matter outside of the narrowly logical part of the consequence relation. In addition to the features of the Ketonen rules that we point out below, one important reason to adopt the Ketonen rules is that they allow us to respect two plausible constraints, which we will illustrate only for conjunction (but which have analogues for disjunction). The first constraint is that if the addition of the premise "Tweety is a penguin" defeats the implication from "Tweety is a bird" to "Tweety can fly," then replacing the premise "Tweety is a bird" with the conjunctive premise "Tweety is a bird, and Tweety is a penguin" should not imply "Tweety can fly." The additive conjunction left-rule would not allow us to respect this first constraint. The second constraint is that it can happen, for instance, that "Tweety is a bird" implies "Tweety can fly" and "Tweety is a plastic toy" implies "Tweety is inanimate," but the combination of the premises "Tweety is a bird" and "Tweety is a plastic toy" does not imply "Tweety can fly, and Tweety is inanimate." However, the multiplicative conjunction right-rule would not allow us to respect this constraint. By using the (quasi) Ketonen-style rules, we can respect both constraints. Thus, our rules allow us to remain classical within the narrowly logical part of the consequence relation, while also allowing us to respect the two just stated constraints outside of the narrowly logical part of the consequence relation.

The second familiar feature of the rules of NMMS is that in moving towards the root on any branch of a proof tree, the sequents never get logically less complex[15] and never fail to include a sentence (embedded or otherwise) that already occurred in any sequent in the branch. Thus, in building a proof tree we compose the root sequent out of all and only the material (that is, atomic sentences) provided by the leaves of the tree. This feature is not only very useful for proving results about sequent calculi, it also lets us see how logically complex sequents derive from our base consequence relation. Moreover, it is an immediate consequence of this feature that the logical extension of any base vocabulary by NMMS is conservative: if all the sentences in Γ and Δ are in our base lexicon $\mathfrak{L}_{\mathfrak{B}}$, then $\Gamma \mid\!\sim \Delta$ just in case $\Gamma \mid\!\sim_{\mathfrak{B}} \Delta$.

Fact 3. *The extension of any consequence relation* $\mid\!\sim_{\mathfrak{B}}$ *to* $\mid\!\sim$ *by* NMMS *is a conservative extension: if* $\Gamma \cup \Delta \subseteq \mathfrak{L}_{\mathfrak{B}}$, *then* $\Gamma \mid\!\sim \Delta$ *just in case* $\Gamma \mid\!\sim_{\mathfrak{B}} \Delta$. *(Appendix, Proposition 26)*

This is an important fact from the perspective of logical expressivism because if logical vocabulary has the function of making explicit reason relations in a base vocabulary, then its introduction into the language should not add or subtract any reason relations that hold between sentences of the base language. Logical vocabulary should make explicit and not change the reason relations with which we started. That our way of adding logical vocabulary is a conservative extension of our base consequence relation ensures this.

The third familiar feature of NMMS is that the rules are invertible (see Appendix, Proposition 27). That means that we can move from the bottom sequent of a rule application to any of the top sequents. If, for instance, we know that $A \vee B \succ C$ is derivable, then we may conclude that $A \succ C$ is derivable, by applying the inverted rule [L$\vee$]. This use of the rule is not itself a derivation, but it is nevertheless the case that the bottom sequent of an application of [L$\vee$] is derivable only if each top sequent is derivable. This is usually expressed by saying that the inverted rule is *admissible* but not *derivable*. All the rules of NMMS are invertible in this sense. The invertibility of our rules has many consequences, such as the following, of which we already know the first two as conditions of adequacy for expressivist logical theories from the previous chapter:

Deduction-Detachment (DD) Condition on Conditionals:
$\Gamma \mid\!\sim A \rightarrow B$ if and only if $\Gamma, A \mid\!\sim B$.

Incoherence-Incompatibility (II) Condition on Negation:
$\Gamma \mid\!\sim \neg A$ if and only if $\Gamma, A \mid\!\sim$ if and only if $\Gamma \# A$.

Antecedent-Adjunction (AA) Condition on Conjunctions:
$\Gamma, A \wedge B \mid\!\sim \Delta$ if and only if $\Gamma, A, B \mid\!\sim \Delta$.

Succedent-Summation (SS) Condition on Disjunctions:
$\Gamma \mid\!\sim A \vee B, \Delta$ if and only if $\Gamma \mid\!\sim A, B, \Delta$.

Thanks to the invertibility of the rules of NMMS, our theory underwrites all of these conditions. As we will see in the next section, that these conditions all hold in our theory is one sense in which the logical vocabulary that is introduced by the rules of NMMS allows us to make explicit reason relations.

To sum up, given any base consequence relation that obeys Containment, its logical extension includes all classically valid implications, it is a conservative extension of the base consequence relation, and it obeys the Deduction-Detachment, the Incoherence-Incompatibility, the Antecedent-Adjunction, and the Succedent-Summation Conditions.

3.1.4 Radical Novelties

In the previous subsection, we have seen that our logical extensions of base consequence relations yield results that are familiar in many ways. Let us now turn to the ways in which our theory is new and unusual—indeed, radically so.

Our account is designed to capture reason relations that are open in the sense that they do not obey structural rules like Monotonicity and Transitivity, that is, the rules of [Weakening] and [Mixed-Cut]. A base consequence relation may, for instance, plausibly include and not include, respectively, the following implications, which are jointly an example of failure of Monotonicity (in which we label premises and conclusions in square brackets as we go along, for future reference)[16]:

(1) [*p*] Tara is a human being. $\mathrel{|\!\!\!\sim_{\mathfrak{B}}}$ [*q*] Tara's body temperature is 37 degrees Celsius.

(2) [*p*] Tara is a human being. [*r*] Tara has a fever. $\not\mathrel{|\!\!\!\sim}_{\mathfrak{B}}$ [*q*] Tara's body temperature is 37 degrees Celsius.

Here (1) is a good implication, but it is defeated by the additional premise that Tara has a fever. We may call such a premise a "defeating premise" or a "defeater." Of course, one might object that (1) is not a good implication because the truth of the premise does not guarantee the truth of the conclusion, as is brought out by (2). Such an objection is, however, merely a way to express the unwillingness to consider the possibility of open reason relations. Once we reject logicism about reasons, it is hard to see what could justify such an unwillingness. So we set this worry aside.

Note that monotonicity may fail not only for implications, or reasons-for, but also for incompatibilities, or reasons-against. Here is an example that we might want to include in our base consequence relation.

(3) [*s*] This figure is a triangle. [*t*] The sum of the inner angles of this figure is larger than two right angles. $\mathrel{|\!\!\!\sim_{\mathfrak{B}}}$

(4) [*s*] This figure is a triangle. [*t*] The sum of the inner angles of this figure is larger than two right angles. [*u*] This figure is a spherical triangle. $\not\mathrel{|\!\!\!\sim}_{\mathfrak{B}}$

Here we have two sentences that are incompatible by themselves, but they become compatible in the presence of the additional sentence that the figure in question is a spherical triangle. We may also express this phenomenon by saying that the incoherence of the set of the first two sentences is cured by the addition of the third sentence.

Moreover, such examples of nonmonotonicity can be turned into examples in which transitivity fails.[17] In particular, if we look at a sentence that is close in its meaning to the negation of the defeating premise and also close to the defeated conclusion, we can construct counterexamples to transitivity by considering these sentences as conclusions. For our purposes the sentences "Tara does not have a fever" and "Tara is healthy" are close enough to allow us to construct the following example:

(5) [*p*] Tara is a human being. $\mathrel{|\!\sim}_{\mathfrak{B}}$ [*q*] Tara's body temperature is 37 degrees Celsius.

(6) [*p*] Tara is a human being. [*q*] Tara's body temperature is 37 degrees Celsius. $\mathrel{|\!\sim}_{\mathfrak{B}}$ [*v*] Tara is healthy.

(7) [*p*] Tara is a human being. $\not\mathrel{|\!\sim}_{\mathfrak{B}}$ [*v*] Tara is healthy.

If we were allowed to apply [Mixed-Cut], we could derive (7) from (5) and (6). But while (5) and (6) are intuitively good implications, it seems implausible that if we are committed to accept that Tara is a human being, we cannot be entitled to reject that Tara is healthy, that is, (7) is intuitively not a good implication. Hence, we have an intuitive counterexample to [Mixed-Cut].

We can also apply this strategy to failures of monotonicity among incompatibilities, if we have sentences that behave closely enough to negations of the sentences in our example of monotonicity failures. Here is how we can apply this to the example (3)-(4) above:

(8) [*s*] This figure is a triangle. $\mathrel{|\!\sim}_{\mathfrak{B}}$ [*w*] The sum of the inner angles of this figure is equal to two right angles.

(9) [*s*] This figure is a triangle. [*w*] The sum of the inner angles of this figure is equal to two right angles. $\mathrel{|\!\sim}_{\mathfrak{B}}$ [*x*] This figure is a Euclidean plane triangle.

(10) [*s*] This figure is a triangle. $\not\mathrel{|\!\sim}_{\mathfrak{B}}$ [*x*] This figure is a Euclidean plane triangle.

Perhaps an opponent would want to say that (10) is indeed a good implication because a triangle is by definition a plane figure and a spherical triangle is, hence, not a triangle. Or an opponent might object that as long as we count spherical triangles as triangles, implication (8) is not a good implication. Of course, it is useful in mathematics to have exact definitions that work in ways that underwrite such objections. But that just means that it is useful in mathematics to have exact definitions that allow our reason relations to be monotonic. We wholeheartedly agree with that. Nothing here hangs on any particular example, and the way we use "triangle" in our example is the way in which spherical triangles are neither excluded nor prominent enough in the concept expressed by "triangle" to undermine the defeasible implication (8), which we take to be the everyday use of the term that is regimented in exact ways in mathematics.[18]

The aspect of our theory that is probably most radical and new is that we take examples like those just given at face value and include them in our consequence relations. We do not try to reconstruct them, on the basis of a monotonic logic, with the help of default rules or partial orderings of models, like prominent approaches to nonmonotonic logics do. Rather, we take the collection of all implications among atomic sentences, those of the examples just given included, as our base consequence relation and we, thus, treat them as axioms of NMMS.

As a toy example, we can do this for the implications above. To do so, we treat the sentences p,...,x as the only atomic sentences of our base lexicon, and we say that our base consequence relation are exactly the implications described as good above plus all instances of Containment. We may call this base $\mathfrak{T}$ (for "toy") and specify it thus:

$$\mathfrak{L}_{\mathfrak{T}} = \{p, q, r, s, t, u, v, w, x\}$$
$$\underset{\mathfrak{T}}{\mid\!\sim} = \{\langle\{p\},\{q\}\rangle, \langle\{s,t\},\emptyset\rangle, \langle\{p,q\},\{v\}\rangle, \langle\{s\},\{w\}\rangle,$$
$$\langle\{s,w\},\{x\}\rangle\} \cup \{\langle\Gamma,\Delta\rangle : \Gamma\cup\Delta \subseteq \mathfrak{L}_{\mathfrak{T}} \text{ and } \Gamma\cap\Delta \neq \emptyset\}$$

The part of the definition of $\underset{\mathfrak{T}}{\mid\!\sim}$ that comes after the first union sign ensures that $\underset{\mathfrak{T}}{\mid\!\sim}$ obeys Containment. So, it follows from our observations above that the logical extension of this base includes all classically valid inferences (over the language that results from adding the logical lexicon to $\mathfrak{L}_{\mathfrak{T}}$). In addition to these classically valid implications, however, $\underset{\mathfrak{T}}{\mid\!\sim}$ and its logical extension $\mid\!\sim$ include the implications that correspond to (1), (3), (6), (8), and (9) above. And because the extension is conservative, neither consequence relation includes the implications that correspond to (2), (4), (7), or (10).

In addition to the implications of our base consequence relation, the logically extended consequence relation includes implications of logically complex sentences that make explicit the good implications of (1), (3), (6), (8), and (9) above, which we may label with the subscript e for "explicitation": (1_e) $\mid\!\sim p \to q$, (3_e) $\mid\!\sim \neg(s \wedge t)$, ($6_e$) $\mid\!\sim (p \wedge q) \to v$, ($8_e$) $\mid\!\sim s \to w$, (9_e) $\mid\!\sim (s \wedge w) \to x$. The sentences on the right in these implications are all theorems of the logically extended consequence relation, in the sense of being implications of the empty set. None of them is, however, a theorem of classical logic. Rather, that these sentences are theorems of the extended consequence relation makes explicit what implies what and what is incompatible with what in our base consequence relation. For example, (1_e) makes explicit, in the form of a theorem, that p implies q. And (3_e) makes explicit, in the form of a theorem, that s and t are incompatible.

Let us highlight, as a side remark, that what matters here is not the status of these sentences as theorems. As will become clearer below, these sentences could play their explicitating role equally well in the context of other premises and conclusions. And more generally, the notion of a theorem, as a sentence that follows from the empty set, loses much of its interest once we allow that the sentence may not follow from other premises. The more interesting notion is that of a sentence that follows from all premise-sets. Given a base that obeys Containment, this is true in NMMS of all the theorems of classical logic; but a logical extension can also include other sentences of which this is true. We may call such sentences "persistent theorems." We will return to the issue of persistence in the third section of this chapter. At present, we can ignore it and simply note that the theorems in the logically extended vocabulary that are listed above reflect material reason relations that hold in our base consequence relation.

Many familiar principles fail outside of the narrowly logical part of the logically extended consequence relation. The examples from above show that [Weakening] (MO) and [Mixed-Cut] (CT) fail:

MO-failure-1: $p \mid\!\sim q$. But $p, r \not\mid\!\sim q$.

MO-failure-2: $s, t \mid\!\sim$. But $s, t, u \not\mid\!\sim$.

CT-failure-1: $p \mid\!\sim q$. And $p, q \mid\!\sim v$. But $p \not\mid\!\sim v$.

CT-failure-2: $s \mid\!\sim w$. And $s, w \mid\!\sim x$. But $s \not\mid\!\sim x$.

A moment's reflection shows that what is sometimes called "meta-*modus-ponens*" can also fail in the logical extensions of base consequence relations. Meta-*modus-ponens* says that if ϕ is a theorem and $\phi \to \psi$ is a theorem, then ψ is a theorem. Now, we can change our examples of failures of CT by replacing the premise of the first implication by the empty set, which yields sequents of the form $\mid\!\sim \phi$, and $\phi \mid\!\sim \psi$. But $\not\mid\!\sim \psi$. Since $\phi \mid\!\sim \psi$ if and only if $\mid\!\sim \phi \to \psi$, cases of this schematic form are cases in which meta-*modus-ponens* fails. Note, moreover, that such failures can occur despite the fact that, as a classically valid schema, all instances of *modus ponens* hold, that is, all instances of $\phi, \phi \to \psi \mid\!\sim \psi$ hold and, indeed, they are all indefeasible, that is, all results of applications of [Weakening] to such instances also hold. In other words, if the premises of a *modus ponens* inference are merely implied, then the conclusion of the *modus ponens* does not follow; but if the premises are explicitly contained in the premises, then the conclusion follows.

In order to highlight how radically our theory diverges from traditional theories outside of the narrowly logical part of the extended consequence relations, let us look at one last example that combines failures of CT, meta-*modus-ponens*, and distribution principles as substitution rules. Suppose we have a base consequence relation that obeys Containment and that includes the following implications: $\mid\!\sim_{\mathfrak{B}} p$; $\mid\!\sim_{\mathfrak{B}} q, r$; $\mid\!\sim_{\mathfrak{B}} p, q, r$. Let us also stipulate, however, that it is not the case that $\mathrel{|\!\!\sim_{\mathfrak{B}}} p, r$. These implications allow us to construct the following proof tree.

$$\dfrac{\succ p \qquad \dfrac{\succ q, r}{\succ q \vee r}[R\vee] \qquad \dfrac{\succ p, q, r}{\succ p, q \vee r}[R\vee]}{\succ p \wedge (q \vee r)}[R\wedge]$$

Hence, we have $\mathrel{|\!\!\sim} p \wedge (q \vee r)$. Furthermore, the following is a classical valid implication and hence part of our logically extended consequence relation: $p \wedge (q \vee r) \mathrel{|\!\!\sim} (p \wedge q) \vee (p \wedge r)$ and $\mathrel{|\!\!\sim} (p \wedge (q \vee r)) \rightarrow ((p \wedge q) \vee (p \wedge r))$. However, our extended consequence relation does not include the sequent $\mathrel{|\!\!\sim} (p \wedge q) \vee (p \wedge r)$. To see this, consider what a proof tree of this sequent would look like. We would have to derive $\succ (p \wedge q), (p \wedge r)$, which requires in turn that we derive $\succ p, (p \wedge r)$. And a derivation of that sequent would have to look as follows:

$$\dfrac{\succ p, p \qquad \succ p, r \qquad \succ p, p, r}{\succ p, p \wedge r}[R\wedge]$$

Since the middle top sequent of this rule application contains only atomic sentences, it holds just in case $\mathrel{|\!\!\sim_{\mathfrak{B}}} p, r$. But we know that $\not\mathrel{|\!\!\sim_{\mathfrak{B}}} p, r$. Hence, $\not\mathrel{|\!\!\sim} (p \wedge q) \vee (p \wedge r)$. This example illustrates, firstly, that CT and meta-*modus-ponens* can fail in our logically extended consequence relation, as we have already seen above. Secondly, the example illustrates that replacing classically equivalent sentences for one another in an implication can turn a good implication into a bad implication. For $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ are classically equivalent sentences, but while $\mathrel{|\!\!\sim} p \wedge (q \vee r)$ holds in our example $\mathrel{|\!\!\sim} (p \wedge q) \vee (p \wedge r)$ does not hold. And to fully appreciate this point, note that the sentences are not just classically equivalent but they also imply each other in the extended consequence relation in which their substitution as conclusions fails.[19]

To sum up, we have seen in the previous subsection that the logical extensions of base consequence relations that result from applying the rules of NMMS yield results that are familiar and unsurprising in many ways, for instance, by including all classically valid implications. In this subsection, we have seen how radically these logically extended consequence relations diverge from standard treatments of logical vocabulary. These consequence relations can encode open reason relations and, therefore, they are substructural, allowing for failures of Monotonicity and Transitivity. They include implications and theorems that are material in the sense of not being logically valid or logically true, respectively. Principles like meta-*modus-ponens* can fail in such extended consequence relations, even though *modus ponens* holds indefeasibly. And substituting sentences that imply each other in classical logic (and so also in our logical extensions) for one another in an implication can turn a good implication into a bad one, that is, such substitutions are not guaranteed to preserve consequence. All of these unfamiliar and radical features of our logically extended consequence relations are a result of including nonlogical, material implications in these consequence relations, in particular substructural material implications. In other words, all of these features are consequences of the ability to codify material and open reason relations.

3.2 Making Reason Relations Explicit

In the previous section, we have seen how we can extend reason relations among sentences of an atomic base language to reason relations among the logical extension of that language. Thus, we have seen how logical vocabulary can be elaborated from arbitrary material reason relations. This was the first part of the central claim of logical expressivism, namely the claim that logical vocabulary is universally LX. The second part—the X-part—of this claim is that logical vocabulary allows its users to make explicit the reason relations that hold in the base vocabulary as well as those that hold in the logically extended vocabulary. In this section, we want to clarify what it means for a vocabulary to allow us to make reason relations explicit. Doing this will also give us occasion to introduce some extensions and variants of the sequent calculus NMMS.

3.2.1 The Idea of Making Reason Relations Explicit

The basic idea of making reason relations explicit is to turn something that one acknowledges in practice into something that one acknowledges explicitly in what one says. It is a genus of the species of explicitation that takes as its starting point a practical ability, such as cooking or playing a musical instrument, and yields as its result sentences that convey the norms or rules that govern the exercise of the abilities, such as a recipe for a dish or instructions for playing an instrument. If one can cook well or play a musical instrument well, one has the ability to acknowledge the norms or rules of cooking or playing the instrument in one's practice by non-accidentally acting in accordance with them. When one formulates this practical knowledge of the norms or rules as sentences of a recipe or instructions, one acknowledges them explicitly. One makes these norms or rules explicit. Someone can have the ability to acknowledge such norms or rules in practice without having the ability to make them explicit in the form of principles. And someone might be able to convey such norms or rules explicitly while being much less able to acknowledge them in the practical sense of acting non-accidentally in accordance with them.

The ability to make explicit what one acknowledges in one's practice is crucial for engaging in critical reflection and discussion of what the correct norms or rules governing the activity in question are. If we want to engage in critical reflection and discussion about what it is to cook well or to play an instrument well, for instance, we must be able to make explicit what someone who cooks well or plays an instrument well acknowledges practically in what they do.

According to logical expressivism, there is a particular species of the genus of explicitation for which the ability at issue is the ability to assess and respond to reasons for and against claims, which we may call the ability to practically appreciate reason relations. This species of explicitation takes as its starting point the ability to practically appreciate reason relations and yields as its result sentences that convey the norms or rules that govern the practice of practically appreciating reason relations. It makes explicit reason relations. It gives us sentences that allow us to convey what is a reason for or against what. And just as making the abilities to cook or to play an instrument explicit allows us to engage in critical reflection and discussion about exercises of the abilities to cook or play an instrument, so making explicit reason relations allows us to engage in critical reflection and discussion about exercises of the ability to practically appreciate reason relations. According to logical expressivism, it is the essential function of logical vocabulary to provide us with sentences that allow us to do this, that is, sentences that make explicit what is a reason for or against what. Using logical vocabulary, reason relations of implication and incompatibility can be made explicit in claims, and so can themselves be challenged and defended.

One special feature of the species of explicitation that uses logical vocabulary is that the activity of practically appreciating reason relations and the activity of engaging in critical reflection and discussion regarding reason relations are the same kind of activity. For engaging in critical reflection and discussion is one way to engage in practically appreciating reason relations. Hence, we are here dealing with the possibility that exercises of our conceptual abilities make explicit the norms or rules that govern exercises of these very conceptual abilities. In this sense, the explicitation that is at issue in logic is a case in which our conceptual abilities are directed at themselves; it is a kind of self-consciousness.

A related second special feature of this species of explicitation is that it happens in the same language in which the explicitated practical appreciation of reason relations takes place. When we make reason relations explicit by using logical vocabulary, we do not use an overt meta-language in which we make explicit reason relations among sentences of an object-language. When we assert, for instance, the sentence $A \to B$ and treat it as undeniable (relative to a given context), we are not saying that A is a reason for B (relative to the context). Rather, we say that $A \rightarrow B$; but what we do in saying this is to commit ourselves to A being a reason for B (relative to the context). Thus, logical expressivism must not be misunderstood as primarily being a claim about what logically complex sentences say. Rather, it is primarily a claim about what we can do with logically complex sentences, namely to make explicit reason relations. This essential role of logically complex sentences constrains their inferential roles. And according to semantic inferentialism, we can understand the content of logically complex sentences in terms of their inferential roles. Nevertheless, according to logical expressivism, logically complex sentences do not mention but use their constituent sentences; and to assert a logically complex sentence is not to describe some reason relation as holding, but to endorse the reason relation that the sentence makes explicit.

Moreover, according to logical expressivism, logic has a special status among all intellectual pursuits in general and in philosophy in particular. Logic is universal in its scope in that (ideally) it allows us to make explicit any reason relations whatsoever. And these reason relations are what we appreciate in practice in any intellectual pursuit. To be self-conscious about any intellectual pursuit is to be able to engage in critical reflection and discussion about it. So, the ability to engage in logic and use logical vocabulary is an aspect of our self-consciousness; it is the ability to engage in critical reflection and discussion regarding any intellectual pursuit whatsoever. Philosophy is not only crucially concerned with what is a reason for or against what in many particular domains, but critical reflection and discussion of the reason relations that are practically appreciated in doing philosophy are essential to philosophy itself. In this sense, philosophy is an essentially self-conscious intellectual pursuit. And logic studies the vocabulary by which we can manifest this kind of self-consciousness. Thus, logic has a special position with respect to philosophy in particular, because it studies and enables us critically to control the use of the vocabulary that allows us to engage in philosophy self-consciously, that is, to do philosophy with the ability to engage in critical reflection and discussion regarding philosophy. And this ability is an essential part of philosophy, and indeed of any sophisticated intellectual pursuit at all.

We can now see that our question about what it means to make reason relations explicit is really asking what we need in order to engage in critical reflection and discussion regarding reason relations, and so the meanings of our nonlogical words. We argued in the previous chapter that at a minimum this requires that we have a conditional that obeys DD and a negation that obeys II. And we saw in the previous section that the logical vocabulary introduced by NMMS meets these desiderata. We can now spell out the idea of explicitation that lies behind DD and II. It turns out that this can be done in two different ways. We discuss the first of these ways in the next subsection and then turn to the second way to spell out the idea.

3.2.2 Explicitation by Implication

Let us look at a schematic example of a case where the explicitating function of logical vocabulary becomes relevant. Suppose my, of course fallible, ability to practically appreciate reason relations includes a disposition to practically treat A as a reason for B and to treat A together with B as a reason against C. Thus, we may say that in my exercises of the ability to appreciate reason relations, I act as if $A \mid\!\sim B$ and as if $\{A, B\}\#C$, that is, as if $A, B, C \mid\!\sim$. Suppose that neither of us is willing to accept or reject any of A, B, or C. However, you are not willing to treat A, supposing we accepted it, as a reason for B, nor are you willing to treat A together with B as a reason against C. For you, these three sentences have nothing to do with one another, as far as reason relations go. Finally, suppose that we have (and take ourselves to have) shared background commitments, namely we accept everything in Γ and we reject everything in Δ. What would we need in order to start to discuss and reflect on our disagreement?

Note that we cannot engage in any critical discussion or reflection by wondering about A or B or C directly. For we want to engage in critical reflection about what we would have reason to accept or reject before and independently of engaging in any particular ground-level commitments. We would have some way forward, however limited it may be, if there were a sentence such that treating this sentence as undeniable covaried with treating A as a reason for B and if there was another sentence such that treating that sentence as undeniable covaried with treating A together with B as a reason against C. It is easy to see that, according to NMMS, $\Gamma, A \mid\!\sim B, \Delta$ if and only if $\Gamma \mid\!\sim A \rightarrow B, \Delta$; and we have $\Gamma, A, B, C \mid\!\sim \Delta$ if and only if $\Gamma \mid\!\sim \neg(A \wedge B \wedge C), \Delta$. So, the logically complex sentences $A \rightarrow B$ and $\neg(A \wedge B \wedge C)$ fit the bill, respectively. Given my dispositions to acknowledge reason relations in practice, I have to treat these sentences as undeniable, relative to our shared background commitments, whereas you will refuse to treat them as undeniable. If we have the ability to dispute and reflect on whether a sentence is undeniable, we now have sentences that can serve as the targets in our disagreement. In this sense, the sentence $A \rightarrow B$ allows us to make explicit that A is a reason for B, and the sentence $\neg(A \wedge B \wedge C)$ allows us to make explicit that C is incompatible with A together with B—and thereby bring these reason relations within reach of our critical practices of challenging and defending claims.

In order to serve as the target of our dispute, what matters is not whether you or I accept $A \rightarrow B$. Rather, what matters is whether we treat rejecting it as ruled-out, as out-of-bounds, that is, whether we treat it as if one cannot be entitled to such a rejection. In other words, what matters is whether we treat our background commitments as providing reasons for $A \to B$. I will do so, and you won't. We can now engage, in effect, in a critical discussion about whether A is a reason for B by examining our shared background commitments and highlight the aspects that we take to be relevant to us having reasons for $A \to B$. Similarly, what matters for our ability to engage in a critical debate about whether A together with B is a reason against C is not whether either one of us accepts $\neg(A \wedge B \wedge C)$. Rather, what matters is whether we treat it as undeniable, that is, whether we take our shared background commitments to provide reasons for $\neg(A \wedge B \wedge C)$. I will do so, and you won't.

This is explicitation by implication. It must not be confused with the explicitation *of* implications that we mentioned in the previous chapter, where what is implied by a set of premises is added to these premises as a further premise. Explicitation *by* implication, by contrast, does not involve turning a conclusion into a premise. Rather, in explicitation by implication, we start with a particular reason relation. And according to this conception of explicitation, what it takes to make explicit a particular reason relation is to have a sentence that is undeniable just in case the reason relation indeed holds. That is, the reason relation holds if and only if the sentence is implied by the background commitments, against which we want to assess the particular reason relation at issue. We can summarize the general format of this kind of explicitation of reason relations in the following definition:

Definition 4 (Making Reason Relations Explicit by Implication).
(For) The sentence ϕ makes explicit that Θ is a reason for Λ if and only if, for all Γ and Δ, we have $\Gamma \mid\!\sim \phi, \Delta$ just in case $\Gamma, \Theta \mid\!\sim \Lambda, \Delta$.
(Against) The sentence ϕ makes explicit that Θ is a reason against Λ if and only if, for all Γ and Δ, we have $\Gamma \mid\!\sim \phi, \Delta$ just in case $\Gamma, \Theta, \Lambda \mid\!\sim \Delta$.

It is easy to see that NMMS introduces logical vocabulary that allows us to make explicit by implication any reason relations among finite sets. To see this, let's write $\bigwedge\{x_1, ..., x_m\}$ for $x_1 \wedge ... \wedge x_m$ and $\bigvee\{x_1, ..., x_m\}$ for $x_1 \vee ... \vee x_m$. Now, by DD, AA, and SS above, for any pair of non-empty sets Θ and Λ, it is easy to see that $\Gamma, \Theta \mid\!\sim \Lambda, \Delta$ if and only if $\Gamma \mid\!\sim \bigwedge \Theta \to \bigvee \Lambda, \Delta$. So, for any (finite) Θ and Λ, the sentence $\bigwedge \Theta \to \bigvee \Lambda$ makes explicit that Θ is a reason for Λ, in the sense of (For). Similarly, by AA and II, we have $\Gamma, \Theta, \Lambda \mid\!\sim \Delta$ if and only if $\Gamma \mid\!\sim \neg \bigwedge \Theta \cup \Lambda, \Delta$. Hence, for any finite Θ and Λ, the sentence $\neg \bigwedge \Theta \cup \Lambda$ makes explicit that Θ is a reason against Λ, in the sense of (Against). We can summarize this result in the following fact.

Fact 5. *For any two finite sets of sentences, from the extended language $\mathfrak{L}$ for any base $\mathfrak{B}$, the logically extended lexicon of that base includes sentences that make explicit by implication when the relation of being a reason for and when the relation of being a reason against holds between the two sets. (Appendix, Proposition 28)*

This fact is one sense in which the vocabulary introduced with the help of NMMS is universally explicative of reason relations. For any autonomous base vocabulary, once we extend it to include logical vocabulary, we can make explicit by implication every reason relation among arbitrary finite sets of sentences.

We have now reached our aim of showing that the logical vocabulary of NMMS is universally LX. As shown in the previous section, the logical vocabulary of NMMS can be elaborated from any autonomous base vocabulary. And as just shown, it allows its users to make explicit arbitrary reason relations among sentences of any autonomous base vocabulary and its logical extension. This means that defining logical expressions for a given base vocabulary by the rules of NMMS meets the desiderata of logical expressivism.

We can now see what it means to adopt logical expressivism and to understand logic as the discipline that studies the vocabulary that universally allows us to make explicit what is a reason for and against what. If we use "conceptual self-consciousness" for the ability to be explicit about the exercises of one's own conceptual abilities and this means to make explicit the reason relations that constrain these exercises of conceptual abilities, then logic is the discipline that studies the vocabulary that allows us to manifest conceptual self-consciousness. Logic thus understood studies the tools that allow us to engage in critical reflection and discussion of the appreciation of reason relations of any kind whatsoever, including open reason relations. This is the conception of logic that animates logical expressivism, and the theory that we have developed in this chapter illustrates how one can formulate a logical theory based on this conception of logic. Thus, we have now put on the table a formally rigorous proposal for how to understand logical expressivism, and how to do logic in a way that is informed and guided by logical expressivism.

3.2.3 Explicitation by Sequents

Besides explicitation by implication, there is also a second and related way to spell out what it means to make reason relations explicit. The idea is that logical vocabulary makes reason relations explicit in the sense that any implication that involves logically complex sentences as premises or conclusions holds only if and because some definite implications among only atomic sentences hold. In this subsection, we explain and explore this way to understand the idea that logical vocabulary makes reason relations explicit.

The second way to understand the idea that logical vocabulary makes explicit reason relations is to understand it as saying that all reason relations among logically complex sentences reflect or express reason relations among atomic sentences in a unique way. According to this way of understanding the idea, it is not a sentence that is implied by some premises that makes explicit reason relations. Rather, it is an entire implication that makes explicit particular reason relations among the base vocabulary. The idea is that every derivable sequent is, as it were, a projection of reason relations among sentences of the base vocabulary. On this way of thinking, for instance, the implication $p \mathrel{|\!\sim} q \to r, s$ makes explicit that the base consequence relation includes the implication $p, q \mathrel{|\!\sim} r, s$. The former implication is a projection of the latter; and note that a sequent featuring logically complex sentences might, in this sense, represent a collection of implications in the base. Let's call this "Explicitation by Sequents."

Definition 6 (Making Base Reason Relations Explicit by One Sequent). Let `AtomicImp` $\subseteq \mathrel{|\!\sim}_{\mathfrak{B}}$ be a set of implications in the base consequence relation. Then the sequent $\Gamma \succ \Delta$ makes explicit (by one sequent) the collection of reason relations `AtomicImp` $= \{\Theta_1 \mathrel{|\!\sim}_{\mathfrak{B}} \Lambda_1, \ldots, \Theta_n \mathrel{|\!\sim}_{\mathfrak{B}} \Lambda_n\}$ if and only if ($\Gamma \mathrel{|\!\sim} \Delta$ just in case $\Theta_1 \mathrel{|\!\sim}_{\mathfrak{B}} \Lambda_1$, and ..., and $\Theta_n \mathrel{|\!\sim}_{\mathfrak{B}} \Lambda_n$).

The logical vocabulary of NMMS is such that for every implication in the extended consequence relation, there is a unique set of implications in the base consequence relation that it makes explicit in this sense. For the invertibility of the rules of NMMS ensures that the following holds:

Theorem 7 (*Projection*). *For any sequent* $\Gamma \succ \Delta$, *with* $\Gamma \cup \Delta \subseteq \mathfrak{L}$, *there is a unique set* `AtomicImp` *of base vocabulary sequents such that* $\Gamma \succ \Delta$ *is derivable in* $\mathsf{NMMS}_{\mathfrak{B}}$ *just in case* `AtomicImp` $\subseteq \mathrel{|\!\sim}_{\mathfrak{B}}$. *(Appendix, Proposition 29)*

This means that every implication among sentences in the logically extended vocabulary makes explicit (by one sequent) a collection of reason relations in the base vocabulary. To put it differently, every sequent featuring logically complex vocabulary reflects a unique set of sequents featuring only atomic sentences: the complex sequent holds just in case all the atomic sequents hold.

It follows that for any set, `ComplexImp`, of sequents in the logically extended language, there is a unique set, `AtomicImp`, of sequents in the nonlogical base vocabulary such that all the complex sequents in `ComplexImp` hold if and only if all the base sequents in `AtomicImp` hold. Moreover, this relation does not depend on what other sequents hold in the base or the extended vocabulary. Rather, this relation holds among the sets `ComplexImp` and `AtomicImp` for all base vocabularies and extensions. Hence, the set of complex sequents `ComplexImp` makes explicit (by sequents) the set of base sequents `AtomicImp` independently of what we choose as our base vocabulary. That Projection holds means that the logical vocabulary of NMMS can make arbitrary nonlogical reason relations explicit by sequents, in this sense. This is a second sense, beside explicitation by implication, in which the logical vocabulary of NMMS is universally explicative of reason relations. We take these two senses of being universally explicative—namely the sense of making arbitrary reason relations explicit by implication and by sequents—to be the two senses of being universally explicative that are central to logical expressivism.

We can now ask whether the converse is also true, that is, whether every set of implications in the base vocabulary is represented by a single implication in the logically extended consequence relation. Unfortunately, such a strong representation relation does not hold in NMMS. There are collections of base vocabulary implications that cannot be made explicit by a single sequent in the extended consequence relation. To see why, suppose that $p \mathrel{|\!\sim} q$ and $r \mathrel{|\!\sim} s$ but $p, r \not\mathrel{|\!\sim} q, s$. We can make the first two implications explicit by $\mathrel{|\!\sim} p \to q$ and $\mathrel{|\!\sim} r \to s$. But we cannot combine these two implications into a single implication. For, in order to derive $\succ (p \to q) \wedge (r \to s)$, we would need not only $\succ p \to q$ and $\succ r \to s$ but also $\succ p \to q, r \to s$. And the derivation of the latter sequent would require that $p, r \succ q, s$ is derivable, which it is not because $p, r \not\mathrel{|\!\sim} q, s$.

We could avoid such cases by requiring that base consequence relations are closed under the following rule, which we may call "Mingle-Mix": if $\Gamma \mathrel{|\!\sim} \Delta$ and $\Theta \mathrel{|\!\sim} \Lambda$, then $\Gamma, \Theta \mathrel{|\!\sim} \Delta, \Lambda$. If base consequence relations must obey Mingle-Mix, then it cannot happen that $p \mathrel{|\!\sim} q$ and $r \mathrel{|\!\sim} s$ but $p, r \not\mathrel{|\!\sim} q, s$. Unfortunately, Mingle-Mix is not plausible in a nonmonotonic setting. It seems plausible, for instance, that "The patient has symptom X" implies "The patient has disease XX" and "The patient has symptom Y" implies "The patient has disease YY"; but "The patient has symptoms X and Y" does not imply "The patient has disease XX or YY" (because the combination perhaps strongly suggests that the patient has disease ZZ). However, if we want to allow that Mingle-Mix fails in material consequence relations, then there are collections of implications in base consequence relations that cannot be made explicit by one single sequent in NMMS.

While this is a limitation of NMMS as formulated above, we can tweak our theory in such a way that our logical vocabulary is explicative of any collections of reason relations by one sequent. To do so, we must reject Contraction, that is, the principle that $\Gamma, \phi \mathrel{|\!\sim} \Delta$ just in case $\Gamma, \phi, \phi \mathrel{|\!\sim} \Delta$ and that $\Gamma \mathrel{|\!\sim} \phi, \Delta$ just in case $\Gamma \mathrel{|\!\sim} \phi, \phi, \Delta$. Hence, we must work with multi-sets of sentences instead of sets of sentences. If we do that, we get the desired result if we change our sequent rules to the more familiar Ketonen-style rules, which are the following (marking the tweaked connectives by adding a bar above them):

$$\frac{\Gamma \succ \Delta, A \qquad B, \Gamma \succ \Delta}{\Gamma, A \bar{\rightarrow} B \succ \Delta}\,[\mathrm{L}\bar{\rightarrow}] \qquad \frac{\Gamma, A \succ B, \Delta}{\Gamma \succ A \bar{\rightarrow} B, \Delta}\,[\mathrm{R}\bar{\rightarrow}]$$

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \bar{\wedge} B \succ \Delta}\,[\mathrm{L}\bar{\wedge}] \qquad \frac{\Gamma \succ \Delta, A \qquad \Gamma \succ \Delta, B}{\Gamma \succ \Delta, A \bar{\wedge} B}\,[\mathrm{R}\bar{\wedge}]$$

$$\frac{\Gamma, A \succ \Delta \qquad \Gamma, B \succ \Delta}{\Gamma, A \bar{\vee} B \succ \Delta}\,[\mathrm{L}\bar{\vee}] \qquad \frac{\Gamma \succ \Delta, A, B}{\Gamma \succ \Delta, A \bar{\vee} B}\,[\mathrm{R}\bar{\vee}]$$

The negation rules and the rules with just one top sequent remain the same as in NMMS. We call the sequent calculus that results from this change $\mathsf{NMMS}^{\setminus ctr}$. To see that the logical vocabulary of $\mathsf{NMMS}^{\setminus ctr}$ can make explicit by implication arbitrary particular reason relations, it suffices to note that DD, II, AA, and SS all hold in $\mathsf{NMMS}^{\setminus ctr}$. Hence, the logical vocabulary of $\mathsf{NMMS}^{\setminus ctr}$ can make individual reason relations explicit by implications in the same way as the vocabulary of NMMS. However, in $\mathsf{NMMS}^{\setminus ctr}$ we can combine the results of such explicitation into a single sequent. For example, because of $[\mathrm{R}\bar{\wedge}]$, if $\succ p \bar{\rightarrow} q$ and $\succ r \bar{\rightarrow} s$ are both derivable, then so is $\succ (p \bar{\rightarrow} q) \bar{\wedge} (r \bar{\rightarrow} s)$, and vice versa. Thus, the limitation regarding explicitation of collections of reason relations by sequents that arose for NMMS does not arise for $\mathsf{NMMS}^{\setminus ctr}$. And in general, the logical vocabulary of $\mathsf{NMMS}^{\setminus ctr}$ is such that, for any (finite) collection of reason relations among (finite) sets, there is a sequent that makes exactly those reason relations explicit.[20]

Proposition 8. *For any finite set,* `AtomicImp`, *of sequents among atomic sentences, there is a sequent* $\Gamma \succ \Delta$, *with* $\Gamma \cup \Delta \subseteq \mathfrak{L}$, *such that* $\Gamma \succ \Delta$ *is derivable in* $\mathsf{NMMS}_{\mathfrak{B}}^{\setminus ctr}$ *just in case* `AtomicImp` $\subseteq \mathrel{|\!\sim}_{\mathfrak{B}}$.

This result allows us to formulate a representation theorem for $\mathsf{NMMS}^{\setminus ctr}$. For we can now see that in $\mathsf{NMMS}^{\setminus ctr}$ every finite set, `AtomicImp`, of implications in the base consequence relation is represented by a single sequent in the logically extended consequence relation, and every implication in the logically extended consequence relation represents a unique set of implications in the base consequence relation. Moreover, there is a unique set, `ExtImp`, of sequents in the logically extended consequence relation that represent a given set, `AtomicImp`, of implications in the base consequence relation. So, for any sequent in `ExtImp` it holds just in case all the sequents in `AtomicImp` hold. This yields the following representation theorem:

Theorem 9 (*Representation*). *For every finite set,* `AtomicImp`*, of sequents in the base vocabulary of* $\mathfrak{L}_{\mathfrak{B}}$ *there is a set of sequents,* `ExtImp`*, in the extended vocabulary of* $\mathfrak{L}$ *introduced by* $\mathsf{NMMS}_{\mathfrak{B}}^{\setminus ctr}$*, such that, for every sequent in* `ExtImp`*, it holds if and only if all the sequents in* `AtomicImp` *hold.*

This theorem tells us that the logical vocabulary introduced by $\mathsf{NMMS}^{\setminus ctr}$ is explicative of reason relations in the sense that not only does every sequent featuring logical vocabulary reflect a collection of reason relations in the base consequence relation but also every finite collection of reason relations in the base vocabulary can be represented in a single sequent that features logical vocabulary.

The price one has to pay in order to have logical vocabulary that is universally explicative in the stronger sense of making arbitrary finite collections of reason relations explicit and, hence, underwriting the representation theorem just stated is that one must reject contraction and, thus, allow that whether or not some premises imply some conclusions may depend on how many times a premise or conclusion occurs in the implication. We will mostly work with NMMS and assume contraction (by working with sets), thus being content with the explicative power of Projection without Representation. We will, however, occasionally discuss the option of rejecting contraction and use $\mathsf{NMMS}^{\setminus ctr}$ instead of NMMS.

Moreover, we should acknowledge that there is another sense in which the logical vocabulary of NMMS may be thought to fail to make explicit reason relations. For, one may distinguish between weak and strong representation of the reason-for relation by saying that ϕ weakly represents that Θ is a reason for Λ if and only if, for all Γ and Δ, we have $\Gamma \mathrel{|\!\sim} \phi, \Delta$ just in case $\Gamma, \Theta \mathrel{|\!\sim} \Lambda, \Delta$. But ϕ strongly represents that Θ is a reason for Λ if and only if, for all Γ and Δ, we have $\Gamma \mathrel{|\!\sim} \phi, \Delta$ just in case $\Gamma, \Theta \mathrel{|\!\sim} \Lambda, \Delta$ and, also, $\Gamma \mathrel{|\!\sim} \neg\phi, \Delta$ just in case $\Gamma, \Theta \not\mathrel{|\!\sim} \Lambda, \Delta$. Given this terminology, the conditional of NMMS and $\mathsf{NMMS}^{\setminus ctr}$ represents the reason-for relation only weakly and not strongly. For, it is not the case that $\Gamma \mathrel{|\!\sim} \neg(\bigwedge \Theta \to \bigvee \Lambda), \Delta$ if and only if $\Gamma, \Theta \not\mathrel{|\!\sim} \Lambda, \Delta$. In an analogous way, the negation of conjunctions only weakly represents the reason-against relation. For, it is not the case that $\Gamma \mathrel{|\!\sim} \neg\neg \bigwedge \Theta \cup \Lambda, \Delta$ if and only if $\Gamma, \Theta, \Lambda \not\mathrel{|\!\sim} \Delta$. For the purposes of our project in this book, two quick remarks about this issue must suffice. (i) The idea of strongly representing the reason-for relation is fraught with difficulties. The idea obviously requires, for instance, that, for every set and every conditional, the set implies either the conditional or its negation. But it seems wrong to think that we always either have reason for an arbitrary conditional or for its negation. (ii) The issue is related in interesting ways to the topic of validity predicates and the so-called v-Curry Paradox, and we think that the issue is best discussed in that context, which is different from our present context (see Beall and Murzi, 2013).[21] So, we set aside the idea of strongly representing reason relations.

This concludes our discussion of how logical vocabulary makes explicit reason relations. Let us take stock. We saw that the logical vocabulary of NMMS can make explicit arbitrary instances of the relation of being a reason-for and being a reason-against, in the sense of explicitation by implication. The logical vocabulary of NMMS can do this for any base reason relations between sets of sentences whatsoever, including nonmonotonic and nontransitive reason relations. In this sense, the logical vocabulary is universally explicative of reason relations. Moreover, the logical vocabulary of NMMS is such that every implication in the extended vocabulary reflects a unique set of implications in the base vocabulary. This is a second sense in which the logical vocabulary of NMMS makes reason relations explicit. However, the logical vocabulary of NMMS cannot represent arbitrary collections of implications in the base vocabulary in a single implication in the logically extended vocabulary. We can move to logical vocabulary that allows us to do this by rejecting Contraction and changing our sequent calculus to $\mathsf{NMMS}^{\backslash ctr}$. This yields an interesting representation theorem for the logical vocabulary of $\mathsf{NMMS}^{\backslash ctr}$. Overall, we have seen in the previous section that the logical vocabulary of NMMS can be elaborated from any autonomous base vocabulary. And we have seen in this section that the logical vocabulary of NMMS can make explicit arbitrary reason relations of any autonomous base vocabulary. Taking these results together, we have shown that the logical vocabulary of NMMS is universally LX. It is the kind of vocabulary that can play the role that is the essential and characteristic role of logical vocabulary, according to logical expressivism.

3.3 Making Local Structure Explicit

We have repeatedly emphasized the importance of allowing for failures of structural principles, especially failures of monotonicity and transitivity. Unless a theory allows for such failures, it cannot codify open reason relations. And if logical expressivism is right in claiming that the characteristic function of logical vocabulary is to make explicit reason relations in general, including open reason relations, then any such theory cannot provide an account of what logic makes explicit. Such a theory might correctly specify the narrowly logical part of a consequence relation, but it cannot account for the essential explicitating function of logical vocabulary that shows up in the capacity of logically complex sentences that are not logically true to codify material reason relations.

Someone might acknowledge this importance of allowing for failures of structural principles while also insisting that structural principles hold in some areas of any consequence relation worth theorizing and that it is of crucial importance to our practice of giving and asking for reasons what these areas are. We agree. After all, we sometimes seem to tacitly assume that the reason relations that govern a particular discourse are transitive and monotonic, and this seems to happen in particular in very sophisticated and successful kinds of discourse, like those in mathematics, some parts of physics, and the like. Moreover, sometimes we try to exploit only the reason relations of classical logic and to codify all other reason relations in the form of explicit definitions, which we then use as premises. In light of such phenomena, it would be unsatisfying merely to be told that the usual structural rules are invalid and should be rejected. In this section, we show how we can move beyond such an unsatisfying position. In particular, we show how we can theorize structural features that hold locally in particular regions of consequence relations, and how we can introduce logical vocabulary that makes such local structural features explicit in the object language.

3.3.1 Regions of Monotonicity

In general, we hold that a good implication, $\Gamma \mid\!\sim \Delta$, may be defeated by the addition of a further premise, such that $\Gamma, A \mid\!\not\sim \Delta$. And we hold that this can also happen when we add further conclusions, such that it might be that $\Gamma \mid\!\not\sim B, \Delta$. However, there will typically be a wide range of additions to an implication that do not defeat the implication. A pair of sets of sentences $\langle X, Y \rangle$ belongs to this range of non-defeating additions to $\Gamma \mid\!\sim \Delta$ if and only if $\Gamma, X \mid\!\sim Y, \Delta$, that is, just in case adding the first member to the premises and the second member to the conclusions yields another good implication. We call the set of such pairs of non-defeating additions the "range of subjunctive robustness" of an implication.[22]

Definition 10 (Range of subjunctive robustness, $\texttt{RSR}\langle \Gamma, \Delta \rangle$)**.** Given a set of premises, Γ, and a set of conclusions, Δ, the range of subjunctive robustness of $\langle \Gamma, \Delta \rangle \in \mathcal{P}(\mathfrak{L}) \times \mathcal{P}(\mathfrak{L})$, written $\texttt{RSR}\langle \Gamma, \Delta \rangle$, is the set of pairs, $\langle X, Y \rangle$, such that $\Gamma, X \mid\!\sim Y, \Delta$; that is, $\texttt{RSR}\langle \Gamma, \Delta \rangle = \{\langle X, Y \rangle \in \mathcal{P}(\mathfrak{L}) \times \mathcal{P}(\mathfrak{L}) : \Gamma, X \mid\!\sim Y, \Delta\}$.

We can add any premises and conclusions to a good implication without defeating the implication, as long as the addition is in the implications range of subjunctive robustness. If we want to be explicit about an implication's range of subjunctive robustness, we can write $\Gamma \mathrel{|\!\!\sim}^{\uparrow R} \Delta$ for $\forall \langle X, Y \rangle \in R$ $(\Gamma, X \mathrel{|\!\!\sim} Y, \Delta)$. Then $\Gamma \mathrel{|\!\!\sim}^{\uparrow R} \Delta$ if and only if $R \subseteq \texttt{RSR}\langle \Gamma, \Delta \rangle$, and $\texttt{RSR}\langle \Gamma, \Delta \rangle$ is the largest set, Z, for which $\Gamma \mathrel{|\!\!\sim}^{\uparrow Z} \Delta$.[23] The larger the range of subjunctive robustness of an implication is, the more it approximates a sequent that holds persistently, that is, the more it approximates a sequent where monotonicity holds locally.

When the range of subjunctive robustness of an implication is maximal it includes all pairs of sets of sentences. If this is the case for $\Gamma \mathrel{|\!\!\sim} \Delta$, then any application of [Weakening] yields another good implication, that is, for any $\langle X, Y \rangle \in \mathcal{P}(\mathfrak{L}) \times \mathcal{P}(\mathfrak{L})$ we have $\Gamma, X \mathrel{|\!\!\sim} Y, \Delta$. We could write this as $\Gamma \mathrel{|\!\!\sim}^{\uparrow \mathcal{P}(\mathfrak{L}) \times \mathcal{P}(\mathfrak{L})} \Delta$, but to avoid clutter we will write it simply thus: $\Gamma \mathrel{|\!\!\sim}^{\uparrow} \Delta$. Hence, monotonicity holds locally at the sequent $\Gamma \mathrel{|\!\!\sim} \Delta$ just in case $\Gamma \mathrel{|\!\!\sim}^{\uparrow} \Delta$. Given that we think that monotonic consequence is an important and noteworthy part of consequence, we can now express this by saying that we are not only interested in $\mathrel{|\!\!\sim}$ but also in $\mathrel{|\!\!\sim}^{\uparrow}$, that is, not only in all good implications but also those implications at which monotonicity holds locally.

According to the conception of logic formulated in logical expressivism, logical vocabulary should ideally allow us to make explicit when monotonicity holds locally at an implication. With this interest in mind, the following facts about our sequent rules are noteworthy: Firstly, if the range of subjunctive robustness of a sequent includes all possible additions of atomic sentences, then it includes all possible additions of any sentences whatsoever. Or, in more formal terms:

Fact 11. *$X, \Gamma \succ \Delta, Y$ is derivable for all $\langle X, Y \rangle \in \mathcal{P}(\mathfrak{L}_{\mathfrak{B}}) \times \mathcal{P}(\mathfrak{L}_{\mathfrak{B}})$ if and only if $Z, \Gamma \succ \Delta, U$ is derivable for all $\langle Z, U \rangle \in \mathcal{P}(\mathfrak{L}) \times \mathcal{P}(\mathfrak{L})$; that is, $\mathcal{P}(\mathfrak{L}_{\mathfrak{B}}) \times \mathcal{P}(\mathfrak{L}_{\mathfrak{B}}) \subseteq \texttt{RSR}\langle \Gamma, \Delta \rangle$ if and only if $\mathcal{P}(\mathfrak{L}) \times \mathcal{P}(\mathfrak{L}) \subseteq \texttt{RSR}\langle \Gamma, \Delta \rangle$. (Appendix, Proposition 30)*

In other words, if we can weaken an implication with arbitrary sets of atomic sentences, then monotonicity holds locally at that implication. Hence, if we can keep track of when we can weaken a sequent with arbitrary sets of atomic sentences, then we can keep track of where monotonicity holds locally.

The second noteworthy fact is that our sequent rules are such that if all the top sequents of an application of a rule hold persistently, then the bottom sequent also holds persistently:

Fact 12. *If all the top sequents of an application of any rule of* NMMS *(or* $\mathsf{NMMS}^{\backslash ctr}$*)*[24] *are good implications at which monotonicity holds locally, then the bottom sequent is a good implication at which monotonicity holds locally, that is, if for every top sequent* $\Gamma \succ \Delta$ *of an application of a rule of* NMMS *(or* $\mathsf{NMMS}^{\backslash ctr}$*)* $\Gamma \mathrel{|\!\sim}^{\uparrow} \Delta$ *holds, then, for the bottom sequent* $\Theta \succ \Sigma$*, we have* $\Theta \mathrel{|\!\sim}^{\uparrow} \Sigma$*. (Appendix, Proposition 31)*

Given these two facts, we can introduce a new kind of sequent arrow, $\succ^{\uparrow}$, such that $\Gamma \succ^{\uparrow} \Delta$ is an axiom of our sequent calculus just in case for all $\langle X, Y \rangle \in \mathcal{P}(\mathfrak{L}_{\mathfrak{B}}) \times \mathcal{P}(\mathfrak{L}_{\mathfrak{B}})$ we have $X, \Gamma \mathrel{|\!\sim}_{\mathfrak{B}} \Delta, Y$. Moreover, we now stipulate that all the rules of NMMS apply if all the top sequents are of the new kind, and then a bottom sequent of the new kind can be derived. For example, if our base obeys Containment, we will have $p \succ^{\uparrow} p$. And we can then apply the rule [R¬] to derive the sequent $\succ^{\uparrow} p, \neg p$. Similarly, [R∨] now lets us derive $\succ^{\uparrow} p \vee \neg p$. By Fact 11, we know that $p \mathrel{|\!\sim}^{\uparrow} p$. And using Fact 12 we can infer that $\mathrel{|\!\sim}^{\uparrow} p \vee \neg p$. Hence, the sentence $p \vee \neg p$ follows monotonically—or persistently—from every set of premises. If we call the sequent calculus with the changes just specified $\mathsf{NMMS}^{\uparrow}$, we can formulate this insight in general terms as follows:

Proposition 13. *The sequent* $\Gamma \succ^{\uparrow} \Delta$ *is derivable in* $\mathsf{NMMS}^{\uparrow}$ *if and only if monotonicity holds locally at* $\Gamma \mathrel{|\!\sim} \Delta$*, that is,* Γ *implies* Δ *persistently. (Appendix, Proposition 32)*

Note that the monotonic part of our consequence relation may include implications that are not in the narrowly logical part of the consequence relation. We may, for example, include the following implication in our base consequence relation: “PP is a parent of CC.” $\mathrel{|\!\sim}$ “CC is a child of PP.” Moreover, we may want to include arbitrary applications of [Weakening] to this implication with sentences from our base lexicon in our base consequence relation. If we do that, this implication will hold monotonically in our logically extended consequence relation. However, the implication is not part of the narrowly logical part of our extended consequence relation because its derivation requires axioms that are not instances of Containment.

As logical expressivists, we don’t just want a concept of implications at which monotonicity holds in our metalanguage. We would ideally also like to be able to make this fact explicit in our object language. More specifically, we would like to be able to make explicit, in the object language, that something is not just implied but that it is implied persistently. Persistence means the implication holds no matter what the additional context might be, that is, come what may. That is one clear sense of "necessarily." If our premise is that PP is a parent of CC, then this does not just imply that CC is a child of PP. Rather, it implies that necessarily CC is a child of PP; it implies that, relative to our premise, CC cannot fail to be a child of PP. Hence we may write the object-language expression that we want to add as "$\Box$"—like the necessity operator in standard modal logics, which we will call the "Monotonicity Box." The idea is to use this operator to mark implications that have a maximal range of subjunctive robustness. So that "PP is a parent of CC" implies: "$\Box$ CC is a child of PP."

We can therefore expand our language $\mathfrak{L}$ to $\mathfrak{L}^{\Box}$ by including sentences in which $\Box$ occurs as a one-place connective, in the usual way. And let us add the following rules to $\mathsf{NMMS}^{\uparrow}$ and call the resulting calculus $\mathsf{NMMS}^{\Box}$, where square brackets around an upward arrow indicate that the upward arrow is optional, that is, the bottom sequent is derivable with and also without the upward arrow.

$$\frac{\Gamma, A \succ \Delta}{\Gamma, \Box A \succ \Delta}\,[\mathrm{L}\Box] \qquad\qquad \frac{\Gamma \succ^{\uparrow} A, \Delta}{\Gamma \succ^{[\uparrow]} \Box A, \Delta}\,[\mathrm{R}\Box]$$

The left-rule for the box ensures that $\Box A$ implies everything that A implies. And the right-rule says that if A is implied in a sequent decorated with an upward arrow, then $\Box A$ is implied. Continuing our example from above, it is easy to see that $\succ \Box(p \vee \neg p)$ is derivable in $\mathsf{NMMS}^{\Box}$, given that the base obeys Containment. In general, the following holds:

Proposition 14. *The sequent* $\Gamma \succ \Box A, \Delta$ *is derivable in* $\mathsf{NMMS}^{\Box}$ *if and only if* $\Gamma \mathrel{|\!\!\sim}^{\uparrow} A, \Delta$. *(Appendix, Proposition 33)*

This result captures the sense in which the Monotonicity Box allows us to make explicit local areas in which monotonicity holds. For it allows us to make explicit when something is not just implied but persistently implied. We can thus not only theorize monotonic consequence, including material monotonic consequence, in our meta-language but we can make this important structural feature of reason relations explicit in our object language. Moreover, just like the results above, this holds universally, that is, it holds for any base vocabulary whose reason relations we may want to make explicit.

It is worth noting that while our Monotonicity Box behaves like a standard necessity operator in many ways, it also differs in significant ways from such standard necessity operators. Among its familiar behavior is that, given any base that obeys Containment, all of the following implications hold, using "$\Diamond$" to abbreviate "$\neg\Box\neg$": $\Box A \mathrel{|\!\!\sim} \Diamond A$, and $\neg\Box A \mathrel{|\!\!\sim} \Diamond\neg A$; moreover, (K) $\Box(A \to B) \mathrel{|\!\!\sim} (\Box A \to \Box B)$, and (4) $\mathrel{|\!\!\sim} \Box(A \to A)$, and (B) $\mathrel{|\!\!\sim} A \to \Box\Diamond A$, and (D) $\mathrel{|\!\!\sim} \Box A \to \Diamond A$, and (5) $\mathrel{|\!\!\sim} \Diamond A \to \Box\Diamond A$ all hold.

Although these results look familiar from the modal logic S5, there are important differences. The necessitation rule is not sound in $\mathsf{NMMS}^{\Box}$ because it can happen that $\succ A$ is derivable but $\succ \Box A$ isn't derivable (and it perhaps should not be derivable because $\mid\!\sim^{\uparrow} A$ doesn't hold). Furthermore, some results would seem very strange for familiar modal logics, such as that the following implications hold, given a base that obeys Containment: $\mid\!\sim A \to \Box A$ and $\neg\Box A \mid\!\sim \neg A$. Although these results might seem strange from the perspective of standard modal logics, they are what one should expect, given that the Monotonicity Box makes explicit local monotonicity. For example, $\mid\!\sim A \to \Box A$ makes explicit by implication that A monotonically implies A, which is ensured by Containment. And $\neg\Box A \mid\!\sim \neg A$ holds because $\mid\!\sim \neg A, \Box A$, which holds because $\mid\!\sim^{\uparrow} \neg A, A$, which is in turn ensured by Containment and our negation rules.

To sum up, we have shown how to introduce a modal operator that makes explicit where monotonicity holds locally, that is, the conclusion $\Box A$ is implied just in case the conclusion A is implied persistently. If our base consequence relation obeys Containment, then all theorems of classical logic will be persistent theorems of our extended consequence relation. So, if ϕ is a theorem of classical propositional logic, then we have $\mid\!\sim^{\uparrow}\phi$ and, hence, $\mid\!\sim \Box\phi$. Thus, we can make explicit by implication that the theorems of classical logic are persistently implied by any set of premises. Moreover, whenever the implication $\Gamma \mid\!\sim \Delta$ is indefeasible, we have $\mid\!\sim^{\uparrow} \bigwedge\Gamma \to \bigvee\Delta$ and, hence, $\mid\!\sim \Box(\bigwedge\Gamma \to \bigvee\Delta)$. So we can make explicit by implication any implication that holds indefeasibly—saying of it in the logically extended object language that it holds indefeasibly.

3.3.2 Classicality, Contraction, and the General Case

In the previous subsection, we showed how one can make explicit when monotonicity holds locally at an implication. The strategy that we used there can be generalized to many other cases. To illustrate this, let us look at classical logic again.

We have acknowledged above that there are discourses in which classical logic is of crucial importance and all other implications are wrapped in definitions of non-logical terms that one can use as premises. In this kind of discourse, critical reflection and discussion may require that we can make explicit implications that hold in classical logic. As already intimated above, these are all and only the implications that can be derived in NMMS from axioms that are instances of Containment. Those are the only implications in a base consequence relation that hold in classical propositional logic. Now, the analogue of Fact 12 above holds for classicality, namely:

Fact 15. *If all the top sequents of an application of any rule of* NMMS *hold in classical propositional logic, then so does the bottom sequent.*

In parallel to what we did above, we can introduce a new kind of sequent arrow, $\succ^{cl}$, and say that $\Gamma \succ^{cl} \Delta$ is an axiom of our new calculus just in case $\Gamma \cup \Delta \subseteq \mathfrak{L}_{\mathfrak{B}}$ and $\Gamma \cap \Delta \neq \emptyset$, that is, if it is an atomic instance of Containment. Moreover, we add the one-place operator $(\!|cl|\!)$ to our language, and we add the following rules to NMMS, thus yielding a calculus we call $\mathsf{NMMS}^{(\!|cl|\!)}$, where the square brackets indicate that the "$[cl]$"-mark in the bottom sequent is optional:

$$\frac{\Gamma, A \succ \Delta}{\Gamma, (\!|cl|\!)A \succ \Delta}\ [\mathrm{L}(\!|\mathrm{cl}|\!)] \qquad\qquad \frac{\Gamma \succ^{cl} A, \Delta}{\Gamma \succ^{[cl]} (\!|cl|\!)A, \Delta}\ [\mathrm{R}(\!|\mathrm{cl}|\!)]$$

It is then easy to prove the following result:

Proposition 16. *If all the sentences in* $\Gamma \cup \Delta \cup \{A\}$ *are in the language of classical propositional logic, then the sequent* $\Gamma \succ (\!|cl|\!)A, \Delta$ *is derivable in* $\mathsf{NMMS}^{(\!|cl|\!)}$ *if and only if* $\Gamma \vdash_{CL} A, \Delta$. *(Appendix, Proposition 34)*

We can thus make explicit the classicality of an implication in the same generic way in which we can make explicit that an implication holds monotonically. As already intimated above, there can be monotonic implications that are not classical implications. Moreover, there can be monotonic implications that are not transitive, whereas classical implications are closed under Cut. So, the Monotonicity Box and the classicality operator are genuinely distinct notions.

Let us now step back and consider the overall strategy at work here. The general recipe behind the operator that makes monotonicity explicit and the operator that makes classicality explicit is the following. (a) There is a feature some implications in our base vocabulary have. (b) This feature is preserved by the rules of NMMS, that is, if all the top sequents in a rule application have the feature, then so does the bottom sequent. (c) An implication in the logically extended language has the feature only if the corresponding sequent is derivable from just those axioms that have the feature. If (a)–(c) hold, we can introduce a new sequent arrow whose axioms are the base sequents with the feature at issue. We can read all the sequent rules as applying to the new kind of sequent arrow, and we can add rules that allow us to add a new operator on the left without further ado and allow us to add the operator on the right if the top sequent uses the new kind of sequent arrow. The result is an operator that makes explicit implications in the logically extended vocabulary that have the feature at issue.

This recipe can be applied in many cases. We have seen how it applies in the cases of monotonicity and classicality. It can also be applied to Contraction, which can fail in $\mathsf{NMMS}^{\setminus ctr}$, as explained above. So let us run through the general recipe again for this case. Let us use $\mathbb{F}$ to talk about the relevant feature, in our case Contraction. For the case of contraction, we can define our feature as follows: $\mathbb{F}(\Gamma \mathrel{|\!\sim} \Delta)$ just in case $\Gamma \mathrel{|\!\sim} \Delta$ and, for all $\Theta \succ \Sigma$, if $\Theta \succ \Sigma$ can be derived from $\Gamma \succ \Delta$ by just applications of Contraction, then $\Theta \mathrel{|\!\sim} \Sigma$.

(a) Some implications, $\Gamma \mathrel{|\!\sim}_{\mathfrak{B}} \Delta$ in our base are such that, for all $\Theta \succ \Sigma$, if $\Theta \succ \Sigma$ can be derived from $\Gamma \succ \Delta$ by just applications of Contraction, then $\Theta \mathrel{|\!\sim}_{\mathfrak{B}} \Sigma$. Those are the base implications such that $\mathbb{F}(\Gamma \mathrel{|\!\sim}_{\mathfrak{B}} \Delta)$.

(b) Given the rules of $\mathsf{NMMS}^{\setminus ctr}$, if all the top sequents of an application of a rule have feature $\mathbb{F}$, then the bottom sequent has feature $\mathbb{F}$. That is, if for all the top sequents it is admissible to apply the Contraction rule, then it is admissible to apply the Contraction rule to the bottom sequent.

(c) If some sequent derivable in $\mathsf{NMMS}^{\setminus ctr}$ has feature $\mathbb{F}$, then it is derivable from sequents that have feature $\mathbb{F}$. That is, all sequents for which the application of the Contraction rule is admissible, can be derived from sequents for which the Contraction rule is admissible.

Since (a)–(c) hold for the feature of Contraction being admissible, we can introduce a new sequent arrow, $\succ^f$, and say that $\Gamma \succ^f \Delta$ is an axiom just in case $\mathbb{F}(\Gamma \mathrel{|\!\sim}_{\mathfrak{B}} \Delta)$. We allow the rules of $\mathsf{NMMS}^{\setminus ctr}$ to apply when all top-sequents use the new kind of sequent arrow $\succ^f$. And we add the following rules:

$$\frac{\Gamma, A \succ \Delta}{\Gamma, (\!|f|\!) A \succ \Delta}\,[\mathrm{L}(\!|f|\!)] \qquad\qquad \frac{\Gamma \succ^f A, \Delta}{\Gamma \succ^{[f]} (\!|f|\!) A, \Delta}\,[\mathrm{R}(\!|f|\!)]$$

And we can then show that our new operator makes explicit the feature at issue, which is the admissibility of Contraction in the present case. We can formulate the result in general terms as follows.

Proposition 17. *The sequent* $\Gamma \succ (\!|f|\!) A, \Delta$ *is derivable in* $\mathsf{NMMS}^{\setminus ctr,(\!|f|\!)}$ *if and only if* $\mathbb{F}(\Gamma \mathrel{|\!\sim} A, \Delta)$.

For the particular case of the admissibility of Contraction, this result tells us that our new operator allows us to make explicit when an implication holds in such a way that Contraction can be safely applied.

3.3.3 Cautious Monotonicity and Cumulative Transitivity

With the general strategy from the previous subsection in hand, we may turn to two structural principles that are often deemed desirable in nonmonotonic logics, but which we have argued in the previous chapter are not desiderata for our project. These are Cautious Monotonicity (CM) and Cumulative Transitivity (CT), which can be formulated thus:

$$\frac{\Gamma \succ A \qquad \Gamma \succ \Delta}{\Gamma, A \succ \Delta}\,[\mathrm{CM}] \qquad\qquad \frac{\Gamma \succ A \qquad \Gamma, A \succ \Delta}{\Gamma \succ \Delta}\,[\mathrm{CT}]$$

As already intimated in the previous chapter, CM says that one can never lose consequences by making implications explicit, in the sense of adding them to one's premises. And CT says that one can never gain consequences by making implications explicit in this sense. Thus, these two structural principles together say that explicitation—in the sense of adding implications to one's premises—is inconsequential. We reject this thesis, and we hold that explicitation can be consequential.

Interestingly, local areas in which CM or CT hold cannot easily be made explicit in the way we can make explicit monotonicity, classicality, and the admissibility of contraction. It is instructive to see why this is so. Let us start with CM.

The problem with CM as a global constraint is that, in the context of a conditional that is adequate for logical expressivism, it implies a much stronger condition. Recall that, according to logical expressivism, the conditional must obey the Deduction-Detachment (DD) condition in order to perform its explicitation function adequately. That is, $\Gamma \mathrel{|\!\sim} A \to B, \Delta$ just in case $\Gamma, A \mathrel{|\!\sim} B, \Delta$. That is what it is for the conditional to make the metalinguistic implication turnstile explicit in the logically extended object language. Now, suppose that $\Gamma, B, C \mathrel{|\!\sim}_{\mathfrak{B}} D$ and $\Gamma \mathrel{|\!\sim}_{\mathfrak{B}} A$ but $\Gamma, B, C \not\mathrel{|\!\sim}_{\mathfrak{B}} A$. In the base consequence relation, CM does not require that $\Gamma, A, B, C \mathrel{|\!\sim}_{\mathfrak{B}} D$ and we can, hence, stipulate that $\Gamma, A, B, C \not\mathrel{|\!\sim}_{\mathfrak{B}} D$. For A is not implied by $\Gamma \cup \{B, C\}$; it is merely implied by a proper subset of them, namely Γ. However, it follows from DD that $\Gamma, B, C \mathrel{|\!\sim} D$ just in case $\Gamma \mathrel{|\!\sim} B \to (C \to D)$. We can now apply CM, and this yields $\Gamma, A \mathrel{|\!\sim} B \to (C \to D)$, which by DD entails that $\Gamma, A, B, C \mathrel{|\!\sim} D$. But that contradicts our stipulation above.

If we insist that our conditional obey DD, as logical expressivism suggests, and we insist that the logical extension of our base consequence relation is conservative, then the only way to ensure that CM holds is to enforce a stronger principle, which we call "Weakening with Implications of Subsets" (WIS):

$$\frac{\Gamma \succ A \qquad \Gamma, \Theta \succ \Delta}{\Gamma, \Theta, A \succ \Delta} \text{[WIS]}$$

This principle says that if some premise-set implies a conclusion, say $\Gamma \cup \Theta \mathrel{|\!\sim} D$, and a subset of this premise-set implies something, say $\Gamma \mathrel{|\!\sim} A$, then we can weaken the original inference by adding the implication of the subset of the premises as another premise, $\Gamma \cup \Theta, A \mathrel{|\!\sim} D$. Thus, this principle allows us to weaken any inference with any sentence that is implied by any subset of the premises of the inference.

If a base consequence relation is closed under WIS, then WIS and, hence, CM hold in the logically extended consequence relation. However, WIS is too strong to be plausible. For, we want to allow for situations like the following: "This is a chair" implies "You can sit on this." "This is a chair" and "This is a piece of art in an exhibition" implies "You are not allowed to touch this." However, if we combine the three claims as premises, "This is a chair" and "This is a piece of art in an exhibition" and "You can sit on this", they do not imply "You are not allowed to touch this." After all, there are exhibitions that invite the audience to participate by using the artwork.

To sum up, if one is willing to accept WIS as a constraint on base consequence relations, then our logical extensions obey WIS and, hence, CM. Given the constraints of logical expressivism that logical extensions of reason relations are conservative and the conditional satisfies DD, this is the only way to ensure that CM holds as a global structural principle.

If we don't want to restrict ourselves to bases that obey WIS, we might still want to make explicit when CM holds locally at an implication. However, an interesting question arises here, namely what we mean by "CM holds locally at an implication." One thing one can mean by this is that CM holds at an implication $\Gamma \mathrel{|\!\sim} A$ if and only if any implication whose premises are exactly Γ can be weakened, on the left side, with A.

Definition 18 (CM holding locally). CM holds locally at an implication $\Gamma \mathrel{|\!\sim} A$ just in case $\Gamma \mathrel{|\!\sim} A$ and, for all B, if $\Gamma \mathrel{|\!\sim} B$, then $\Gamma, A \mathrel{|\!\sim} B$.

If we want to make explicit by implication when CM holds locally in this sense, we will try to introduce an operator, $(\!|cm|\!)$, such that $\Gamma \mathrel{|\!\sim} (\!|cm|\!)A$ if and only if CM holds locally at $\Gamma \mathrel{|\!\sim} A$. However, this idea is self-undermining in the following way. Suppose that CM holds locally at $\Gamma \mathrel{|\!\sim} A$ and $\Gamma \mathrel{|\!\sim} B$. Moreover, $\Gamma \mathrel{|\!\sim} C$ but $\Gamma, A, B \mathrel{|\!\not\sim} C$. Since CM holds at $\Gamma \mathrel{|\!\sim} B$, we can weaken the implication of C to get $\Gamma, B \mathrel{|\!\sim} C$. Given how we would like to make explicit when CM holds locally, we will also have $\Gamma \mathrel{|\!\sim} (\!|cm|\!)A$ and $\Gamma \mathrel{|\!\sim} (\!|cm|\!)B$. It follows from these two sequents that $\Gamma, B \mathrel{|\!\sim} (\!|cm|\!)A$. However, this is wrong; we have $\Gamma, B \mathrel{|\!\sim} C$ but $\Gamma, A, B \mathrel{|\!\not\sim} C$. So there is an implication of the premise-set $\Gamma \cup \{B\}$ that cannot be weakened with

A, contradicting what we want $\Gamma, B \mathrel{|\!\sim} (\!|cm|\!)A$ to mean. Hence, while $\Gamma \mathrel{|\!\sim} (\!|cm|\!)B$ holds if we consider just the implications that do not include our new operator $(\!|cm|\!)$, the very introduction of this operator makes it the case that $\Gamma \mathrel{|\!\sim} (\!|cm|\!)B$ no longer holds.

We call this kind of phenomenon an "expressive paradox." And what we mean by this is a situation in which something holds of an implication or a set of implications but when we add to our object language the resources to make explicit that this holds of the implication(s), it no longer holds. The situation is similar to the attempt to introduce a sentence whose utterance would allow one to make explicit that one is currently not uttering any sentence.

While there may be ways to make explicit when CM holds locally that do not give rise to expressive paradoxes, we shall not pursue this project any further here. We want to point out, however, that the existence of expressive paradoxes implies that the project of logical expressivism has limits. Some features of reason relations may be such that they cannot be made explicit without undermining them. And sometimes it might not be possible to make two things both explicit at the same time, so that we have to choose between the ability to make one explicit and the ability to make the other explicit. We are inclined to think that semantic paradoxes like the Liar Paradox and Curry's Paradox can be understood as expressive paradoxes and that seeing semantic paradoxes as particular instances of the wider class of expressive paradoxes might be fruitful. However, we will not pursue that idea in this work.

Finally, let us turn to Cumulative Transitivity (CT). There is no fully satisfying way to make explicit when CT holds in a region of our consequence relation. However, we can make explicit in which regions CT together with Monotonicity hold in the following way (see Hlobil, 2017).

Definition 19 (MOT-base-regions). An MOT-base-region is a subset of $\mathrel{|\!\sim}_{\mathfrak{B}}$ that is closed under [Weakening] and under CT, that is, a subset such that (i) if $\Gamma \mathrel{|\!\sim}_{\mathfrak{B}} A$ and $\Gamma, A \mathrel{|\!\sim}_{\mathfrak{B}} \Delta$, then $\Gamma \mathrel{|\!\sim}_{\mathfrak{B}} \Delta$, and (ii) if $\Gamma \mathrel{|\!\sim}_{\mathfrak{B}} A$, then $\Gamma, B \mathrel{|\!\sim}_{\mathfrak{B}} A$.

If $mot.n$ is the n^{th} MOT-base-region in which we are interested, we introduce a sequent arrow $\succ^{mot.n}$ and let $\Gamma \succ^{mot.n} \Delta$ be an axiom of our new calculus NMMS^{mot} just in case $\Gamma \mathrel{|\!\sim}_{\mathfrak{B}} \Delta$ holds and is in the MOT-base-region $mot.n$. Each MOT-base-region will then have a logical extension, consisting of the sequents derivable from axioms that are all in the MOT-base-region. This logical extension of the MOT-base-region will also be closed under CT and [Weakening]. We can then introduce an operator that allows us to make explicit which implications belong to this logical extension, namely as follows:

$$\frac{\Gamma, A \succ^{mot.n} \Delta}{\Gamma, (\!|mot.n|\!) A \succ^{[mot.n]} \Delta}\ [\text{L}(\!|\text{mot.n}|\!)] \qquad \frac{\Gamma \succ^{mot.n} A, \Delta}{\Gamma \succ^{[mot.n]} (\!|mot.n|\!) A, \Delta}\ [\text{R}(\!|\text{mot.n}|\!)]$$

We can then show that $\Gamma \mid\!\sim (\!|mot.n|\!) A, \Delta$ holds just in case $\Gamma \mid\!\sim A, \Delta$ is in the logical extension of the MOT-base-region *mot.n*. Hence, if $\Gamma \mid\!\sim (\!|mot.n|\!) A, \Delta$, then [Weakening] and CT are admissible for this implication if the second top premise of the CT application is also in the logical extension of the MOT-base-region *mot.n* (see Hlobil, 2017).

We thus have a way to keep track, in the object language, of regions of our consequence relations in which Monotonicity and Cumulative Transitivity hold, that is, regions that are closed under these rules. This approach could be refined and further developed. For our current purposes, however, it suffices to note that various structural features of consequence relations can be made explicit in the object language. The degree of difficulty and the amount of added complications vary between different structural features. When Monotonicity or classicality hold logically at an implication, we can make this explicit in a straightforward way by using the recipe described in the previous two subsections. This recipe works also when we reject Contraction as a global structural principle and want to make explicit when Contraction is nevertheless admissible for particular sequents. The cases of Cautious Monotonicity and Cumulative Transitivity give rise to more complications. We have thus sketched the beginning of an account of how and to what degree local structural features of consequence relations can be made explicit. What we have said is merely a start in this direction. It suffices, however, to illustrate in what kind of new logical vocabulary a logical expressivist might be interested and how such new logical vocabulary might be introduced. The logical expressivist is interested in making explicit reason relations and their local structural features. And in this section, we have seen how this interest gives rise to questions and projects that have not been addressed, as far as we know, by any extant logical theories.

3.4 Conclusion

The previous chapter introduced a stark distinction between two ways of thinking about the relations between the two topics of our title: logic and reasons. In the simplest terms, logicists about reasons understand reasons in terms of logic: good reasons are, in the end, logically good reasons. Expressivists about logic understand logic in terms of reasons: the defining job of logical vocabulary is to make explicit reason relations of implication and incompatibility. Expressivists must understand the reason relations that logical vocabulary codifies as settled elsewhere, before logic comes on the scene. We have sketched a pragmatics-first order of explanation, according to which those reason relations among sentences are instituted by the norms governing their use in discursive practices of making claims and challenging and defending them. We talk about "reason relations" because being a reason is a relational property, and because reason relations merely constrain and do not determine what one, ultimately, has reason to accept or reject. Indeed, it is a bivalent relational property: reasons are essentially, and not just accidentally, either reasons for or reasons against—hence the two kinds of reason relations: implication and incompatibility. In the next chapter we begin to consider a semantics-first order of explanation, according to which the reason relations are settled by the relations of sentences to worldly states that make them true or false. Also in Chapter Four, we open the discussion of how to understand the relations between those normative pragmatic and representational semantic perspectives on the basic reason relations that expressivists understand as providing the raw materials that metalogical machinery then puts into explicit logical shape.

Since its Fregean origins, modern logic has been largely framed, shaped, and conducted within the confines of a logicist philosophical understanding of the enterprise. (We think that Frege himself was a rational expressivist, but the motivation of our project does not turn on that controversial hermeneutic claim.) The question we addressed in this chapter is how one can and ought to do logic differently, if one instead understands the enterprise philosophically in logical expressivist terms. What kind of logic is motivated by expressivism in the philosophy of logic? How can one best deploy what we have learned technically about logic during its development under logicist auspices, in the service of fulfilling the aspirations of logical expressivism? In subsequent chapters we ask a corresponding question about model-theoretic formal semantics—with special attention to the semantics of logical vocabulary.

In keeping with its logicism, the traditional philosophical understanding of logic focuses on pure logic: reason relations that hold in virtue of logic alone. In the sequent calculus, this means looking only at proof trees whose leaves are instances of Containment, that is, derivations from initial sequents of the form $\Gamma, A \succ A, \Delta$. By contrast, we think logic appears in its most characteristic and revealing guise when it is applied. In the sequent calculus, this means looking at proof trees whose leaves are sequents codifying substantial material implications—those whose goodness is underwritten not by a structural principle such as Containment but rather by the norms governing the use of the nonlogical sentences involved, or the way they represent the world as being.

We can also think of the distinction from the side of analysis rather than synthesis. (Perhaps it seems to be begging the question in favor of logical expressivism to think of things in terms of applying to an antecedent field of material reason relations connective rules that we have stipulated.) Given a set of relations of implication and incompatibility among sets of sentences (including logically complex ones), and a distinction between logical and nonlogical vocabulary, reason relations that hold in virtue of logic alone (those that are due to the "logical form" of the sentences involved rather than their nonlogical content) can be picked out by using the Bolzano-Frege-Quine method of noting invariance under substitution. The logically good implications, are those good implications that remain good under uniform substitution of nonlogical for nonlogical vocabulary (and similarly for incompatibilities), including substitutions that include possible extensions or variations of the nonlogical vocabulary. Substitutionally picking out this subset of purely logical reason relations from an antecedently specified field of reason relations by seeing which ones hold *salva consequentia* only works if one can both tell good from bad implications without appeal to logically good implications, and can demarcate logical from nonlogical vocabulary. Logical expressivism as we have introduced it shows up as an option in the context of an order of explanation that assumes competent linguistic practitioners must have at least a rough and ready capacity to tell what is a reason for and against what—that is, to distinguish successful from unsuccessful rational defenses of and challenges to doxastic commitments (quite apart from whether that feature of the pragmatics is in addition understood as reflecting or instituting representational semantic properties of the sentences involved).

Against this background, logical expressivism then offers its distinctive criterion of demarcation of specifically logical vocabulary. Logical vocabulary is to be distinguished by its characteristic expressive role: making reason relations explicit. Here "making explicit" is formulating relations of implication and incompatibility as the content of declarative sentences, which are understood as what can both be used to defend or challenge other claims by offering reasons for or against them, and can itself be defended or challenged by other claims. In Chapter Two, where the expressivist understanding of the relation between logic and reasons was introduced, it was argued that implicit in this conception of the expressive task distinctive of logical vocabulary is a regulative ideal: logical vocabulary should be elaborated from and explicative of any and every base vocabulary. In our slogan, it should be *universally LX*. Given *any* set of reason relations on a sentential lexicon ("universally"), the rules for introducing logical vocabulary should make it possible to compute ("elaborate"—the "L" in "LX") the reason relations of the logically extended vocabulary entirely from the reason relations of the base vocabulary. Further, when used in accordance with the reason relations elaborated from those of the base vocabulary, the new, logically complex compounds of the original sentences should make it possible to say explicitly ("explicate"—the "X" in "LX") what all those reasons relations are—both those of the base and those of the extended vocabulary. Saying, making explicit, in the relevant sense consists in putting into the form of declarative sentences—that is, in claimable form, as what can be asserted and denied, challenged and defended.

In this chapter, we have introduced the sentential logic that is our best candidate for satisfying this expressive ideal, and shown how it satisfies all the basic criteria of adequacy logical expressivism motivates. Along the way, doing that required substantially clarifying and making more concrete those criteria of adequacy. The metavocabulary we use to introduce logical vocabulary is Gentzen's sequent calculus. It is ideal for our purposes because it treats reason relations (in the form of sequents) as the objects that it manipulates and operates on. Its connective rules, individually and collectively, define functions from reason relations codified in a set of sequents relating sets of sentences of the base vocabulary to reason relations codified in the form of sequents relating sets of sentences of the logically extended vocabulary. The sequent calculus is accordingly explicitly designed, and perfectly suited, to perform the aspect of the function of logical vocabulary that consists in elaborating or computing one set of reason relations from another. It operationalizes the sense in which anyone who knows how to use the base vocabulary to make, challenge, and defend claims expressed by its sentences thereby knows how to do everything they need to know how to do in order to make, challenge, and defend claims expressed by all the sentences of the logically extended vocabulary. From an expressivist point of view, the sequent calculus is the metavocabulary that most perspicuously displays this essential aspect of the fundamental expressive task of logic. In Chapter Five we introduce implication-space semantics as the corresponding ideally perspicuous metavocabulary for computing the conceptual roles played by more complex sentences from those played by simpler ones—where, in accordance with semantic inferentialism, the conceptual roles of sentences are understood as the roles they play in reason relations.

The core of what expressivism in the philosophy of logic asks of logic is contained in the explicitation condition of the "universal LX-ness" formulation. That is the requirement that to qualify as logical, vocabulary must function to make explicit the reason relations of base vocabularies to which it is applied. The fundamental reason relations are implication or consequence, and incompatibility, corresponding to reasons for and reasons against. We began by formulating a precise sense in which familiar logical connectives can count as making such relations explicit, which we denominated as "explicitation by implication." Though they appear in slightly generalized form in the multisuccedent sequent-calculus formulation, explicitation by implication for the basic connectives of sentential logic is epitomized in the two principles governing the core expressive connectives that codify implications and incompatibilities: the Deduction-Detachment (DD) condition on conditionals, which says that $\Gamma \mid\!\sim A \rightarrow B$ if and only if $\Gamma, A \mid\!\sim B$ and the Incoherence-Incompatibility (II) condition on negation, which says that $\Gamma \mid\!\sim \neg A$ if and only if $\Gamma, A \mid\!\sim$. (The auxiliary, aggregative connectives of conjunction and disjunction are governed by the corresponding Antecedent-Adjunction (AA) and Succedent-Summation (SS) conditions.) The DD condition ensures that a context implies the conditional $A \rightarrow B$ just in case, in that context, the antecedent of the conditional A implies its consequent B. That is one clear thing to mean by saying that the conditional codifies in the logically extended object language the implications that are specified by the turnstile in the sequent-calculus metalanguage. In that specific sense, the conditional says that an implication is good. That is explication by implication. Where DD holds, any implication can be explicitly expressed in the form of a conditional. Similar remarks hold of the other connective conditions.

Explicitation, the 'X' portion of "Universal LX-ness," in the sense of explicitation by implication, requires satisfaction of the four conditions DD, II, AA, and SS. The logic NMMS does so. Those conditions are all biconditionals. The metalogical tractability and expressive power of Gentzen's sequent calculus (its enabling the capacity to prove things about what things can be proven), however, depends on requiring that derivations using connective rules always proceed from the logically simpler to the logically more complex. That is, each step in a derivation, from the leaves to the root of a proof tree, involves adding connectives. Gentzen's student Ketonen saw that using the same distinction between derivability and admissibility that is exploited in Gentzen's Cut-eliminability "Hauptsatz" makes possible invertible rules—which permit the sort of biconditionals explicitation by implication requires. For even though derivations always move from the less to the more complex, it can still be arranged that whenever the sequent below the sequent-derivation line is derivable, so are the sequents above the line. Then the converse rule is admissible. That makes it possible to have invertible rules, which if properly chosen can satisfy DD, II, AA, and SS.

The Deduction-Detachment condition, the Incoherence-Incompatibility condition and the other conditions on connective definitions provide sufficient expressive power to guarantee the explicability by implication of any and all individual reason relations. But as we have arranged the logical extension of arbitrary base vocabularies by NMMS, there is also a kind of group or collective explication, which is a further kind of expressive completeness. For any NMMS implication relating sets of logically complex sentences, we can compute exactly what set of implications must hold in any base vocabulary for its logical extension to include that NMMS implication. That is, each implication of one set of logically complex sentences by another codifies some particular set of implications in the base vocabulary. We can understand that as what it says about the reason relations of the base vocabulary. For the logical vocabulary of the non-contractive calculus $\mathsf{NMMS}^{\backslash ctr}$, the converse also holds: every set of implications in the base vocabulary can be made explicit by a single sequent in the logically extended vocabulary. The logic has the expressive power to make explicit in a single implication the fact that any set of implications holds in any base vocabulary. The clearly defined notions of explication by implication, projection, and representation of logical vocabulary relative to the base vocabularies to which it is applied make precise the idea of making reason relations explicit that is at the heart of logical expressivism. We show that NMMS satisfies that expressive ideal: it is LX for any and all base vocabularies.

In sequent calculi, connective rules define functions from reason relations to reason relations within the confines specified by the structural rules. The aspiration to expressive universality of logic with respect to reason relations—its being LX for any antecedent constellation of implications and incompatibilities—would be compromised insofar as the applicability, tractability, and expressive functionality of the connective rules depended on structural presuppositions. The stronger those presuppositions (Monotonicity and Transitivity as mixed-context Cut, or only Cautious Monotonicity and Cumulative Transitivity as shared-context Cut, etc.) the greater the infringement of the expressive ideal. We show how to relax those structural constraints—indeed, to eliminate them (though retaining Containment is shown to pay big dividends).[25] Doing that requires carefully tuning the connective rules so that they do not implicitly reimpose structures that have not been explicitly stipulated in the form of global structural rules. Since we hold, contrary to relevant logics, that licensing, for instance, the move from $\Gamma, A \mid\!\sim \Delta$ to $\Gamma, A \wedge B \mid\!\sim \Delta$ is a way to implicitly reimpose a kind of Monotonicity, relaxing the structural constraints in such a careful way requires mixing additive and multiplicative rules for conjunction and disjunction.[26]

In these ways, logical vocabulary can make explicit arbitrary reason relations, according to the theory developed in this chapter. That is, the logical vocabulary that we have introduced is universally explicative of reason relations, including open reason relations. Hence, we have shown how we can introduce logical vocabulary that is universally LX—that is, that can be elaborated from any base vocabulary and make explicit arbitrary reason relations. Crucially, this includes reason relations that are structurally open rather than closed in various senses. Even radically substructural reason relations can still be codified logically: not only those in which Monotonicity and Transitivity fail, but even consequence and incompatibility relations that are hypernonmonotonic, in failing even Cautious Monotonicity.

We have also shown how vocabulary can be introduced that makes explicit where structural features hold locally in a consequence relation, even where those structural constraints do not hold globally. This includes local regions where Monotonicity, Classicality, and Contraction hold. This new vocabulary for marking local regions of sequents where reason relations satisfy various closure-structural conditions qualify as a kind of logical vocabulary by our expressivist criterion of demarcation. For instance, the intensional modal operator marking persistent (monotonic) sequents—a kind of necessity corresponding to a maximal range of subjunctive robustness of an implication—counts as a bit of logical vocabulary, since it makes explicit a feature of reason relations: that not only is an implication good, but so are all the implications related to it by having premise-sets and conclusion-sets that are supersets of the originals. By contrast, for instance, the set-theoretic epsilon does not qualify as a bit of logical vocabulary, since it does not make explicit any feature of reason relations as such. The new operators that we have introduced for codifying the structural properties of reason relations illustrate that logical expressivism gives rise to new interests and research questions, which extant logical theories have not raised or addressed.

Things had to be arranged so that all three of these kinds of expressive-explicative relations—explication by implication of individual sequents, the capacity to express collections of sequents with logically complex sequents, and the explicit marking with sentential operators of local structural features of sequents—continue to work as desired in the absence of global closure structure. That means that even if the base vocabulary is nonmonotonic (or even hypernonmonotonic, in failing Cautious Monotonicity), any set of base sequents is still expressible explicitly as a set of sequents in the logically extended vocabulary—and so, if we assume the bases are finite, in a single sequent relating sentences in the logically extended lexicon. For this to hold, it must be that the reason relations of the logically extended vocabulary are also open-structured. The extended vocabulary is a conservative extension of the base vocabulary, so at least those substructural or open-structured base sequents will remain so. But their open structure will also be reflected in the structure of the sequents that hold between sets of sentences in the logically extended lexicon. The expressive role of the logical connectives requires that the relations of consequence and incompatibility of the logically extended vocabulary do not satisfy closure-structural principles, so as to be able to make explicit the reason relations of base vocabularies that do not exhibit those structures either.

However, the fact that the implications and incompatibilities relating logically complex sentences in the vocabulary that logically extends particular material base vocabularies are substructural does not mean that the purely logical reason relations of those logically extended vocabularies—the implications and incompatibilities that hold in virtue of logic alone—must also be open-structured. We can extract or abstract such purely logical reason relations by an analogue of the Bolzano-Frege-Quine method of noting invariance under substitution referred to just above. Here what corresponds to observing which substitutions of nonlogical for nonlogical vocabulary preserve the goodness of sequents (what can be substituted *salva consequentia*) is to ask which sequents hold no matter what base vocabulary the logical connective rules of the sequent calculus NMMS are applied to. This will amount to only a tiny part of the sequents relating sets of sentences in the logically extended lexicon that hold relative to any particular material base vocabulary. But the reason relations that in this sense hold in virtue of the logical form of sentences, regardless of the content of the nonlogical sentences that appear there, will have the full topological closure structure that Gentzen and Tarski demanded of notions of logical consequence (and incompatibility as formal inconsistency). In other words, what we are offering is not a nonmonotonic (nontransitive, etc.) logic, as the recent tradition has conceived them. It is rather a logic for codifying nonmonotonic (nontransitive, etc.) consequence relations (and reason relations more generally). The expressivist philosophical reconceptualization of the issue—the story about what kind of logic is wanted once one acknowledges that reason relations do not in general have the strong closure structure of traditional purely logical reason relations—accordingly can be seen to have substantial consequences for the logics one crafts in response.

The striking fact is that in spite of it remaining tractable and retaining its substantial expressive power when applied to radically substructural base vocabularies, NMMS is in a clear sense, just classical logic. In the context of full global topological structural rules (Gentzen's Weakening and Cut, as well as Contraction and Contaiment), the mixed additive-multiplicative, reversible Ketonen rules of NMMS just yield classical logic. There are many such formulations of classical logic that are equivalent (in the sense of determining the same purely logical consequence relation) in the presence of those strong topological closure-structural restrictions, but that come apart if those restrictions are relaxed. It turns out that under such open-structured circumstances, the expressive powers of the various logics varies substantially. We have chosen carefully among those once-equivalent, now-diverging specifications of classical logic, to find one that best satisfies expressivist requirements.

3.5 Appendix

It will prove useful to start by introducing the notion of a proof-search.

Definition 20 (Proof-search). A (root-first) proof-search produces a proof-tree from a sequent $\Theta \succ \Sigma$, which is the root of the tree, by recursively applying the following procedure until the process terminates when the proof-tree no longer changes: (i) If $\Gamma \succ \Delta$ is the leaf of a branch of the tree at the current stage and all the sentences in Γ and Δ are atomic, then the branch remains unchanged. (ii) Otherwise, we look for the first complex sentence in $\Gamma \succ \Delta$ (starting on the left, ordering the sentences in Γ and Δ alphabetically) and build the branch up from that leaf by using the appropriate rule of NMMS. For example, we apply the top-to-bottom version of [L$\vee$] (moving upwards in the tree) if the left-most complex sentence in our sequent is a disjunction, thus yielding a fork in the tree with three new leaves that are the three top sequents of our application of [L$\vee$], and similarly for the other cases. (Although we work with sets (and so contraction is built in), we represent the sets in our sequents with the number of copies of sentences that we get by applying this procedure to the given representation of the root, thus treating our sets (in how we represent them) like multi-sets.)

The results of proof-searches for representations of the root sequent that differ in the numbers of copies of sentences do not differ, up to differences in the number of copies of sentences.

Proposition 21. *Proof-searches on* $\Gamma, A \succ \Delta$ *and* $\Gamma, A, A \succ \Delta$ *yield the same results, and the same holds for proof-searches on* $\Gamma \succ A, \Delta$ *and* $\Gamma \succ A, A, \Delta$, *up to differences in the number of sentences in the representations of the sets.*

Proof. If A is atomic, then the proof-search leaves it untouched. If A is complex, it is a conjunction, disjunction, or negation (treating the conditional as defined for simplicity, with $A \to B$ being $\neg A \vee B$). Suppose $A = B \wedge C$. Then applying our procedure with [L$\wedge$] to $\Gamma, B \wedge C \succ \Delta$ yields $\Gamma, B, C \succ \Delta$, and applying it twice to $\Gamma, B \wedge C, B \wedge C \succ \Delta$ yields $\Gamma, B, C, B, C \succ \Delta$. Thus, the resulting set of premises is identical, up to the number of copies of sentences. The cases for [R$\vee$], [L$\neg$], and [R$\neg$] are analogous. For [R$\wedge$], applying our procedure to $\Gamma \succ B \wedge C, \Delta$ yields $\Gamma \succ B, \Delta$ and $\Gamma \succ C, \Delta$ and $\Gamma \succ B, C, \Delta$. Applying the procedure twice to $\Gamma \succ B \wedge C, B \wedge C, \Delta$ yields $\Gamma \succ B, B, \Delta$ and $\Gamma \succ C, B, \Delta$ and $\Gamma \succ B, C, B, \Delta$ and $\Gamma \succ B, C, \Delta$ and $\Gamma \succ C, C, \Delta$ and $\Gamma \succ B, C, C, \Delta$ and $\Gamma \succ B, B, C, \Delta$ and $\Gamma \succ C, B, C, \Delta$ and $\Gamma \succ B, C, B, C, \Delta$. Each of the conclusion sets in these sequents is identical to that of one of the three sequents above, up to differences in the number of copies of sentences. The case for [L$\vee$] is analogous. ■

Proposition 22. *Proof-searches (for finite sequents) terminate, and their results are the same if we change the order of the sentences in* Γ *and* Δ.

Proof. Proof-searches terminate because the root contains finitely many logical connectives, and the children of a node always contain one fewer connective than the parent node.

To show that the order doesn't matter, it suffices to show that for each pair of rules, the order in which they are applied doesn't matter. If we have, for example, $\neg A, B \vee C, \Gamma \succ \Delta$, applying our procedure to the first two sentences yields: $B, \Gamma \succ \Delta, A$ and $C, \Gamma \succ \Delta, A$ and $B, C, \Gamma \succ \Delta, A$. This result is the same whether we use [L$\neg$] first and then [L$\vee$] or the other way around. The same holds for all pairs of rules. Hence, the result of a proof-search is order-independent. ■

Proposition 23. *If all the leaves of a proof tree of* NMMS *can be weakened with any set of atomic sentences, then the root of the proof tree can be weakened with any set of sentences.*

Proof. By induction on the maximal complexity of the sentences with which one weakens the sequent. ■

Proposition 24. *If base* $\mathfrak{B}$ *obeys Containment, then for all* $\Gamma \cap \Delta \neq \emptyset$ *the sequent* $\Gamma \succ \Delta$ *is derivable in* $\mathsf{NMMS}_{\mathfrak{B}}$, *that is, the logical extension of the base also obeys Containment.*

Proof. By induction on the complexity of the most complex sentences in $\Gamma \cup \Delta$. The base case is immediate from the fact that $\mid\!\sim_{\mathfrak{B}}$ obeys Containment. For the induction step it suffices to note that all instances of Containment that feature sentences of complexity of at most $n + 1$ can be derived from instances of Containment that feature sentences whose complexity is at most n. ■

Proposition 25. *If base* $\mathfrak{B}$ *obeys Containment and* $\Gamma \vdash_{CL} \Delta$ *holds in classical propositional logic, where the sentences in* Γ *and* Δ *are in the logical extension of the lexicon of* $\mathfrak{B}$, *then the sequent* $\Gamma \succ \Delta$ *is derivable in* $\mathsf{NMMS}_{\mathfrak{B}}$, *that is,* $\vdash_{CL} \subseteq \mid\!\sim$.

Proof. If [Weakening] and [Mixed-Cut] hold, then the rules of NMMS are equivalent to (the propositional fragment of) Gentzen's familiar LK sequent rules for classical logic. In Gentzen's LK, it is well-known that [Weakening] can be absorbed into the axioms and the axioms can be restricted to atomic sequents, and [Mixed-Cut] can be eliminated by pushing applications of it up the proof tree to the leaves, where [Mixed-Cut] holds because the instances of Containment are closed under [Mixed-Cut]. ■

Proposition 26. *If* $\Gamma \cup \Delta \subseteq \mathfrak{L}_{\mathfrak{B}}$, *then* $\Gamma \mid\!\sim \Delta$ *just in case* $\Gamma \mid\!\sim_{\mathfrak{B}} \Delta$.

Proof. It suffices to note that all the rules of NMMS introduce logically complex sentences into the bottom sequent. ■

Proposition 27. *All the rules of* NMMS *are invertible, that is, the bottom sequent is derivable just in case all the top sequents are derivable.*

Proof. By induction on proof height. We just do the proof for [L→]; the other cases are analogous. Suppose our proposition holds for proofs of height n, and we derive $\Gamma, A \to B \succ \Delta$ is a proof of height $n+1$. If the root comes by [L→], we are done. If the root comes by any other rule, then $A \to B$ occurs on the left in all top sequents, which are derived in proofs of height n. Hence, we can apply our induction hypothesis, which yields three sequents for each top sequent of the last rule application of our original proof tree. And by applying the last rule applied in the original proof tree we can now derive the desired three sequents: $\Gamma \succ \Delta, A$ and $\Gamma, B \succ \Delta$ and $\Gamma, B \succ \Delta, A$. ■

Proposition 28. *If* $\Theta \cup \Lambda \subseteq \mathfrak{L}$ *and the sets are finite, then* $\mathfrak{L}$ *includes sentences that make explicit by implication that* Θ *is a reason for* Λ *and that* Θ *is a reason against* Λ*; i.e., there are sentences,* ϕ *and* ψ*, such that, for all* Γ *and* Δ*, we have* $\Gamma \mid\!\sim \phi, \Delta$ *just in case* $\Gamma, \Theta \mid\!\sim \Lambda, \Delta$*, and we have* $\Gamma \mid\!\sim \psi, \Delta$ *just in case* $\Gamma, \Theta, \Lambda \mid\!\sim \Delta$.

Proof. $\bigwedge \Theta \to \bigvee \Lambda$ meets the condition for ϕ. And $\neg \bigwedge \Theta \cup \Lambda$ meets the condition for ψ. For, the invertibility of the rules of NMMS ensures that $\Gamma \mid\!\sim \bigwedge \Theta \to \bigvee \Lambda, \Delta$ just in case $\Gamma, \Theta \mid\!\sim \Lambda, \Delta$; and $\Gamma, \Theta, \Lambda \mid\!\sim \Delta$ if and only if $\Gamma \mid\!\sim \neg \bigwedge \Theta \cup \Lambda, \Delta$. ■

Proposition 29. *For any sequent* $\Gamma \succ \Delta$*, with* $\Gamma \cup \Delta \subseteq \mathfrak{L}$*, there is a unique set* `AtomicImp` *of base-vocabulary sequents such that* $\Gamma \succ \Delta$ *is derivable in* $\text{NMMS}_{\mathfrak{B}}$ *just in case* `AtomicImp` $\subseteq \mid\!\sim_{\mathfrak{B}}$.

Proof. All the rules of NMMS are invertible. Hence, supposing that $\Gamma \succ \Delta$ contains finitely many sentences of finite length, the process of applying the NMMS rules repeatedly in a proof search on $\Gamma \succ \Delta$ yields a set of atomic sequents in finitely many steps. Since the order in which we apply these (inverted) rules does not matter, this result is unique. Call the set of these atomic sequents `AtomicImp`. Then $\Gamma \succ \Delta$ is derivable in $\mathsf{NMMS}_{\mathfrak{B}}$ just in case `AtomicImp` $\subseteq \mathrel{|\!\sim}_{\mathfrak{B}}$. ■

Proposition 30. *$X, \Gamma \succ \Delta, Y$ is derivable for all $\langle X, Y\rangle \in \mathcal{P}(\mathfrak{L}_{\mathfrak{B}}) \times \mathcal{P}(\mathfrak{L}_{\mathfrak{B}})$ if and only if $Z, \Gamma \succ \Delta, U$ is derivable for all $\langle Z, U\rangle \in \mathcal{P}(\mathfrak{L}) \times \mathcal{P}(\mathfrak{L})$.*

Proof. The right to left direction is immediate from $\mathfrak{L}_{\mathfrak{B}} \subset \mathfrak{L}$. The left to right direction follows from Proposition 23. ■

Proposition 31. *If for every top sequent $\Gamma \succ \Delta$ of an application of a rule of* NMMS *(or* $\mathsf{NMMS}^{\setminus ctr}$*) $\Gamma \mathrel{|\!\sim}^{\uparrow} \Delta$ holds, then, for the bottom sequent $\Theta \succ \Sigma$, we have $\Theta \mathrel{|\!\sim}^{\uparrow} \Sigma$.*

Proof. We can weaken each top sequent with the sentences (on the left and the right) with which we want to weaken the bottom sequent, and derive the desired sequent using the original rule. ■

Proposition 32. *$\Gamma \succ^{\uparrow} \Delta$ is derivable in* $\mathsf{NMMS}^{\uparrow}$ *if and only $\forall \langle X, Y\rangle \in \mathcal{P}(\mathfrak{L}) \times \mathcal{P}(\mathfrak{L})$ $(X, \Gamma \mathrel{|\!\sim} \Delta, Y)$.*

Proof. Left-to-right: Suppose that $\Gamma \succ^{\uparrow} \Delta$. Then there is a $\mathsf{NMMS}^{\uparrow}$ proof tree in which all leaves are of the form $\Theta \succ^{\uparrow} \Lambda$. So we can weaken all leaves with arbitrary atomic sentences. By Proposition 23, it follows that $\forall \langle X, Y\rangle \in \mathcal{P}(\mathfrak{L}) \times \mathcal{P}(\mathfrak{L})$ $(\Gamma \mathrel{|\!\sim} \Delta)$.

Right-to-left: Suppose that $\forall \langle X, Y\rangle \in \mathcal{P}(\mathfrak{L}) \times \mathcal{P}(\mathfrak{L})$ $(X, \Gamma \mathrel{|\!\sim} \Delta, Y)$. Hence, $\forall \langle X, Y\rangle \in \mathcal{P}(\mathfrak{L}) \times \mathcal{P}(\mathfrak{L})$ $(X, \Gamma \succ \Delta, Y)$ is derivable in $\mathsf{NMMS}^{\uparrow}$. In particular, $\forall \langle X, Y\rangle \in \mathcal{P}(\mathfrak{L}_{\mathfrak{B}}) \times \mathcal{P}(\mathfrak{L}_{\mathfrak{B}})$ $(X, \Gamma \succ \Delta, Y)$ is derivable in $\mathsf{NMMS}^{\uparrow}$. In this case, X and Y cannot enter the proof tree anywhere but in the leaves. Since $\langle \emptyset, \emptyset\rangle \in \mathcal{P}(\mathfrak{L}_{\mathfrak{B}}) \times \mathcal{P}(\mathfrak{L}_{\mathfrak{B}})$, we know that $\Gamma \succ \Delta$ is derivable. It follows that, for each leaf of the proof tree for the root $\Gamma \succ \Delta$, we have the leaf $\Theta \succ \Lambda$ itself and also, $\forall \langle X, Y\rangle \in \mathcal{P}(\mathfrak{L}_{\mathfrak{B}}) \times \mathcal{P}(\mathfrak{L}_{\mathfrak{B}})$, the atomic sequent $X, \Theta \succ \Lambda, Y$. Hence, we have $\Theta \succ^{\uparrow} \Lambda$ for each leaf. So, we can simply decorate the proof tree for $\Gamma \succ \Delta$ with an upward arrow on which sequent, and this yields a proof tree for $\Gamma \succ^{\uparrow} \Delta$. ■

Proposition 33. *$\Gamma \succ \Box A, \Delta$ is derivable in* $\mathsf{NMMS}^{\Box}$ *if and only if $\Gamma \mathrel{|\!\sim}^{\uparrow} A, \Delta$.*

Proof. The right to left direction is ensured directly by our rules. $\Gamma \mathrel{|\!\sim}^{\uparrow} A, \Delta$ holds if $\Gamma \succ^{\uparrow} A, \Delta$ is derivable. And we can apply [R$\Box$] to get $\Gamma \succ \Box A, \Delta$.

For the left to right direction it suffices to show that [R$\Box$] is invertible and Proposition 32 still holds for $\mathsf{NMMS}^{\Box}$. The argument from the proof of Proposition 27 can be adjusted to apply to [R$\Box$]. And the argument for Proposition 32 works exactly as for $\mathsf{NMMS}^{\uparrow}$. ■

Proposition 34. *If all the sentences in* $\Gamma \cup \Delta \cup \{A\}$ *are in the language of classical propositional logic, then the sequent* $\Gamma \succ (\!|cl|\!) A, \Delta$ *is derivable in* $\mathsf{NMMS}^{(\!|cl|\!)}$ *if and only if* $\Gamma \vdash_{\overline{CL}} A, \Delta$.

Proof. Suppose that all the sentences in $\Gamma \cup \Delta \cup \{A\}$ are in the language of classical propositional logic. Then $\Gamma \vdash_{\overline{CL}} A, \Delta$ holds just in case there is a derivation of $\Gamma \succ A, \Delta$ in $\mathsf{NMMS}^{(\!|cl|\!)}$ whose leaves are all instances of Containment. This happens just in case there is a derivation of $\Gamma \succ^{cl} A, \Delta$ in $\mathsf{NMMS}^{(\!|cl|\!)}$; and the rules for $(\!|cl|\!)$ cannot be used in such a derivation because $(\!|cl|\!)$ does not occur in the root sequent. And since [R$(\!|cl|\!)$] is invertible (by the same argument as the other rules), this happens just in case there is a derivation of $\Gamma \succ (\!|cl|\!) A, \Delta$. ■

Notes

1 While a discussion of alternative approaches to nonmonotonic consequence would lead us too far afield, we can illustrate our point in the main text with two quick remarks: First, in standard preferential logics, one adds a partial order over classical models and says that $\Gamma \mid\!\sim A$ if all the models of Γ that are minimal in the partial order are models of A (Kraus et al., 1990). Since there is no model of $\bot$, it follows that $\Gamma \mid\!\sim \bot$ if and only if there is no classical model of Γ, that is, if $\Gamma \vdash_{\overline{CL}} \bot$. Thus, standard preferential logics must treat incoherence as classical and indefeasible.

Second, in standard default logics, one adds default rules of the form $\phi : \mathsf{M}\psi / \chi$ to classical logic, which are to be read as saying that given ϕ one can infer χ, as long as it is consistent to assume that ψ (Reiter, 1980). Then extensions of sets of sentences are defined by a procedure that adds the conclusion of a default whose premises are available, when this can be done in a way that yields a classically consistent result, closing the result under classical consequence. Such procedures can vary in their details (for instance, in the ordering which defaults are applied, etc.). And the consequences of a set of sentences are then defined in terms of such extensions; the simplest options are to take either the union (credulous strategy) or the intersection (skeptical strategy) of all such extensions. A classically consistent set of sentences cannot yield a classically inconsistent extension, as classical consistency is the chief constraint on extensions. Hence, the intersection of such extensions is also classically consistent. So, given a skeptical strategy, $\Gamma \mid\!\sim \bot$ if and only if $\Gamma \vdash_{\overline{CL}} \bot$, as in preferential logics. The union of extensions can be classically inconsistent, and this might change if we add extra sentences to the set of sentences with which we begin. So, the credulous strategy can allow for the nonmonotonicity of incoherence. However, it is still classical consistency that is driving and underlying this account, as the inconsistency of the conclusions arises only at the stage where we take the union of the classically consistent extensions.

2 Cross (2003) gives one of the rare discussions of nonmonotonic incompatibility. Let us point out that we avoid Cross's result that $\Gamma, A \mid\!\sim$ if and only if $\Gamma \mid\!\sim A$ yields a collapse into monotonicity because we reject the principle that if $\Gamma \mid\!\sim A$ and $\Gamma \mid\!\sim \neg A$, then $\Gamma \mid\!\sim$. Given our negation rules below, this principle is equivalent to Cut and so we reject it when we reject transitivity. It is worth noting, however, that we accept the related principle that $\Gamma, A, \neg A \mid\!\sim \Delta$, for any Γ, Δ, and A. We thus hold that explicit contradictions are persistently incoherent and imply everything. But if the contradiction is only implicit (in that both A and not-A are implied), then a distinction can still be made between what follows from the premises and what does not.

3 Below they will sometimes be multi-sets, namely when we discuss potential failures of contraction. Multi-sets are like sets except that multi-sets can agree in all their elements and still be distinct because they include different numbers of occurrences (numbers of copies) of those elements, for example, $\{A, A\}$ and $\{A\}$ are the same set but distinct multi-sets. We will flag when such complications become relevant.

4 These are of course merely the best known generalizations, there are also hypersequent calculi and many other generalizations. The generalization to multi-sets is the only one that will become relevant below.

5 We will write interchangeably of accepting sentences and accepting their contents. We will not offer any treatment of expressions that are sensitive to contexts of utterance or the like; so that we can make the simplifying assumption that sentences always express the same content.

6 There are also formulations of bilateralism in the literature that do not work with multiple conclusions (see Rumfitt, 2000; Smiley, 1996). The conception of consequence in these other formulations differs from ours. So we set them aside, although a detailed comparison would be an interesting project.

7 We think of this interpretation of offering a natural connection between our use of sentences and consequence relations. We thus disagree with Steinberger's (2011) claim that semantic inferentialism is in tension with multiple-conclusion calculi.

8 We will sometimes assume that the sets among which consequence relations hold are finite. However, we make this assumption for convenience. We are open to generalizing our notions of consequence, sequents, proof trees (below), and the like to infinite sets of sentences and "proof trees" of infinite height (and breadth). Since taking this complication into account would lead to technical complications that we wish to avoid here (such as the need for transfinite inductions), we make the simplifying assumptions that implications hold among finite sets and that proof trees are finite in height. Of course, this assumption would have substantive implications if we wanted to add arithmetic to our consequence relations. However, we will not consider arithmetic or other theories for which this assumption will be crucial.

9 This approach was first developed in (Hlobil, 2016), but there it was applied in a Set-Formula setting to yield super-intuitionistic logics. Our use here is closer

to (Kaplan, 2018), where the idea is applied in a Set-Set setting to yield super-classical logics. Applications that yield first-order logics and relevance logics have also been developed (Shimamura, 2019, 2017; Hlobil, 2018).

10 Note that the conditional of NMMS is as much an idealization and simplification of the natural language conditional as the so-called "material" conditional. Like the "material" conditional, our $\phi \to \psi$ is equivalent to $\neg\phi \vee \psi$. Hence, denying the conditional "It is raining → The sky is blue" is out-of-bounds if and merely because it is out-of-bounds to assert "It is raining" while denying "The sky is blue." And, by an inverted application (to be discussed shortly) of [L→], if it is out-of-bounds to assert "It is raining → The sky is blue," then it is also out-of-bounds to deny "It is raining" or to assert "The sky is blue" or to do both. Our aim regarding the conditional here is not to offer a plausible formal codification of the natural language conditional but rather to introduce a conditional that can make explicit reason relations.

11 It can easily be shown that negation together with one of the other connectives suffices to define the remaining two connectives, in a way that is broadly analogous to the functional completeness of the connectives in classical logic.

12 We will not speculate about how this might work at a cognitive or pragmatic level, which would concern only particular agents and particular communities. What matters to us is merely that nothing more than an appropriate ability to use the base language is needed, whatever the mechanism is to transform this ability into the ability to use logical vocabulary.

13 In standard technical language: NMMS has the subformula property.

14 Notable exceptions are connexive logics and strict-tolerant logic. We return to the latter below.

15 For our purposes in this book, we can define the complexity of a sequent (or a sentence) as the number of connectives that occur in it. Sometimes a more careful definition as the number of connectives embedded under which the most deeply embedded atomic sentence occurs are more convenient. However, what we say holds for all such definitions of complexity. Hence, we will often not be precise about which notion of complexity we have in mind.

16 We omit quotation marks to avoid clutter. Strictly speaking, what flanks the snake-turnstile are terms for object language sentences and not object language sentences. We trust context to disambiguate between a sentence and its quote-name.

17 Here we use a recipe for turning examples of monotonicity failures into examples of transitivity failures that was developed, in the context of discussing conditionals, by Ryan Simonelli (2022). Related observations can be found in Nair (2019).

18 Lakatos (1976) might be read as giving an example of how a definition of a polyhedron that avoids such failures of monotonicity and transitivity could arise. The simplistic example regarding triangles is alluded to in Lakatos (1976, 24), when student Sigma suggests that there are some mathematical theorems with exceptions and others without exception, such as "the angle sum of all plane triangles is always equal to two right angles." And the passage continues: "Epsilon (to Kappa): Who is this muddlehead? He should learn something about

logic. Kappa (to Epsilon): And about non-Euclidean plane triangles." It seems to us that the temptation to make such objections shows how powerful the song of the siren of Logicism about reasons is. Our project is to sail past such sirens, tied to the mast if necessary, to explore what lies on the other side.

19 This can happen because, in nontransitive consequence relations, mutual implication of two sentences does not entail that the two sentences can be replaced for each other as premises and conclusions *salva consequentia*. The importance of this fact will become clearer in Chapter Five, when we introduce the notion of conceptual content in terms of substitutability *salva consequentia*.

20 The restriction to finite sets of base sequents holds because it might take infinitely many steps to "combine" infinitely many base sequents into one logically complex sequent. If we are willing to allow for proof trees with infinitely many steps in our sequent calculus, then the theorem holds for all sets of implications in the base consequence relation.

21 A validity predicate that strongly represents the reason-for relation in the validity-free fragment of the language is presented by Hlobil (2020).

22 As we will see in later chapters, this idea applies in important ways to "candidate implications" or "bad implications"; their range of subjunctive robustness is the set of pairs whose addition turns these implications into good implications. This idea will become important in the implication-space semantics below. The definition we give here is already general in this way, although we do not stress this here.

23 It is possible to incorporate into the sequent calculus set theoretic manipulations of the set in the place here occupied by R and Z. This is important in the case of super-intuitionistic logics (see Hlobil, 2016). To quickly illustrate the basic idea: If we define the upward arrow by saying that $\Gamma \mathrel{|\!\!\sim}^{\uparrow Z} \Delta$ if and only if $\forall \langle \Theta, \Sigma \rangle \in Z\ (\Theta, \Gamma \mathrel{|\!\!\sim} \Sigma, \Delta)$, then the following are immediate consequences: If $\Gamma \mathrel{|\!\!\sim}^{\uparrow U} \Delta$ and $\Gamma \mathrel{|\!\!\sim}^{\uparrow W} \Delta$, then $\Gamma \mathrel{|\!\!\sim}^{\uparrow U \cup W} \Delta$. And if $U \subseteq \texttt{RSR}$ of $\langle \Gamma, \Delta \rangle$, then $\Gamma \mathrel{|\!\!\sim}^{\uparrow U} \Delta$ and $\Gamma \mathrel{|\!\!\sim}^{\uparrow U \cap V} \Delta$ for any set V. Such consequences can then inform new kinds of sequent rules (see Hlobil, 2016). We will not need this machinery in this work. Hence, we will not introduce any details here.

24 Everything in this section applies to both calculi. To avoid clutter, we will henceforth refrain from mentioning $\mathsf{NMMS}^{\backslash ctr}$ explicitly.

25 Here and elsewhere we set aside the structural principle of Permutation. We consider Contraction only occasionally. Thus, we usually take for granted that collections of premises and conclusions are, in effect, sets.

26 That is not usually done because it blocks showing admissibility of Cut, given failures of Monotonicity. But we do not want that, since not all the base vocabularies whose implication relations we want to make explicit satisfy the strong (mixed-context) transitivity condition that is Cut.